

4. Fermions

4.1 Fields and Lorentz invariance

So far we have discussed the quantisation of a scalar field: (spin 0)

$$\phi(x) \xrightarrow[\text{traj}_0]{\text{Lorentz}} \phi'(x') = \phi(x) \quad (4.1)$$

$$\text{with } x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (4.2)$$

$$(\text{and } \phi'(x) = \phi(\Lambda^{-1} \cdot x))$$

Vector fields: (spin 1)

$$A^\mu(x) \longrightarrow \Lambda^\mu{}_\nu A^\nu(x) \quad (4.3)$$

$$A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1} \cdot x)$$

Tensor fields:

$$F^{\mu\nu}(x) \longrightarrow \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(x) \quad (4.4)$$

In general:

$$\phi^i(x) \rightarrow R(\Lambda)^i_j \phi^j(x) \quad (4.5)$$

↙ representation

with general index i , e.g. $i = \{1, \nu, \nu r, \dots\}$

scalar: $R(\Lambda) = 1$ trivial rep

vector: $R(\Lambda) = \Lambda$ fundamental rep

(2nd rank) tensor: $R(\Lambda) = (\Lambda^\mu_\sigma \Lambda^\nu_\tau)$ tensor rep

Representation $R: G \rightarrow R(G)$, G group:

$$R(\mathbb{1}) = \mathbb{1}; \quad R(g \cdot h) = R(g) \cdot R(h) \quad (4.6)$$

Example: $SO(3)$ rotations in \mathbb{R}^3

(i) trivial rep: $R(\Lambda) = 1$ $\Lambda \in SO(3)$

(ii) fundamental rep: $R(\Lambda) = \Lambda$ Lie group

Lie group: $\Lambda = e^{i\vec{\omega} \cdot \vec{J}}$ with J^1, J^2, J^3

J^i : generators

Lie algebra

$$(4.7)$$

J^i generate infinitesimal rotations

with axis x^i : $(\partial^i x^j = -\delta^{ij})$

$$J^i = -i \varepsilon^{ijk} x^j \partial^k \quad (4.8)$$

$$= -\frac{1}{2} \varepsilon^{ijk} J^{jk}, \quad J^{jk} = -i(x^j \partial^k - x^k \partial^j)$$

Lié-algebra:

$$[J^i, J^j] = i \varepsilon^{ijk} J^k \quad \text{check} \quad (4.9)$$

QM: n -dim repres. of spins with $n=2s+1$

Spin $1/2$: generators $\sigma_{1/2}^i$ with

$$[\sigma_{1/2}^i, \sigma_{1/2}^j] = i \varepsilon^{ijk} \sigma_{1/2}^k \quad (4.10)$$

with Pauli matrices σ :

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spinor representation (4.11)

Remarks:

(i) The Lie algebra provides local information about the Lie group (Tangential space). Important examples:

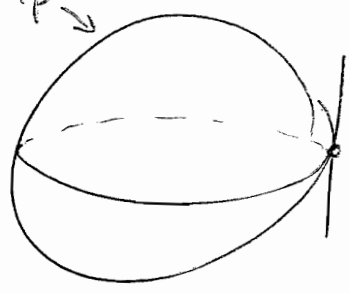
(1) $SO(3)$ and $SU(2) \cong S^3$

Lie algebra: structure constant

$$[t^a, t^b] = i \epsilon^{abc} t^c$$

↑ generators

group →



← algebra

(Lie group = diff. manifold)

$SU(2)$ is universal double covering of $SO(3) \cong RP^3$

(2) $SO(1,3)$ and $SL(2, \mathbb{C})$

$$\Downarrow$$

$$\Lambda_{\mu}^{\nu} = \delta_{\mu}^{\nu} + i T_{\mu}^{\nu} \text{ infinitesimally} \quad (4.14)$$

$$SO(1,3): \Lambda_{\mu}^{\nu} \Lambda_{\rho}^{\sigma} \eta_{\nu\sigma} = \eta_{\mu\rho}$$

$$\Rightarrow (\delta_{\mu}^{\nu} + i T_{\mu}^{\nu}) (\delta_{\rho}^{\sigma} + i T_{\rho}^{\sigma}) \eta_{\nu\sigma} = \eta_{\mu\rho}$$

$$\Rightarrow \boxed{T_{\mu\rho} + T_{\rho\mu} = 0} \quad +O(T^2) \quad (4.15)$$

We conclude that T has $\frac{16-4}{2} = 6$ free para. ^{diag}
antisym

Generators M :

$$\boxed{T_{\mu}^{\nu} = \frac{\omega^{\rho\sigma}}{2} (M_{\rho\sigma})_{\mu}^{\nu}} \quad (4.16)$$

with Lie algebra

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho})} \quad (4.17)$$

Eq. (4.17) is the Lie algebra of $SO(1,3)$ rotations. To see this, we extend the $SO(3)$ -generators of rotations, J^{ij} in eq. (4.8) to boosts (J^{0i}):

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (4.18)$$

The $J^{\mu\nu}$ satisfy (4.17).

General representations: look for M that satisfy (4.17).

E.g. fundamental representation:

$$(M^{\mu\nu})_{\rho\sigma} = i(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) \quad (4.19)$$

Boosts and rotations are given by

$$J_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \quad \text{rotations} \quad (4.20)$$

$$K_i = M_{0i} \quad \text{boosts}$$

Example for boost \mathcal{L} (along x_1 -axis)

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$= \left(e^{\omega K_1} \right)^{\mu}_{\nu} \quad \text{with rapidity } \omega$$

where

$$\omega = \text{artanh } v/2$$

and generator

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The generators in eq. (4.20) make the structure of the Lorentz group apparent:

$$\begin{aligned} [J^i, J^j] &= i \varepsilon^{ijk} J^k \\ [K^i, K^j] &= -i \varepsilon^{ijk} J^k \\ [J^i, K^j] &= i \varepsilon^{ijk} K^k \end{aligned} \quad (4.21)$$

(ii)* $SU(2)$ and $SL(2, \mathbb{C})$ with generators $(J^i + iK^i, J^i - iK^i)$ are universal covering groups of $SO(3)$ and $SO(1,3)$ respectively.

universal covering group \hat{G} of G :

(simply connected group $\hat{G} \ni G'$)