4.2 Spinor fields

Combine Boost K^i and Rotations J^i, (4.13)

into

$$N^i = \frac{1}{2}(J^i + iK^i)$$
$$N^{i+} = \frac{1}{2}(J^i - iK^i) \qquad (4.22)$$

The N's and N^+'s have $SU(2)$ Lie-algo.

$$\boxed{[N_i^{(+)}, N_j^{(+)}] = i\varepsilon_{ijk} N_k^{(+)}} \qquad (4.23)$$

show

$\Rightarrow$ 2-dim spin 1/2 representation:

left-handed $$\Lambda_L = \exp\left\{ \tfrac{i}{2}\sigma^i(\underset{\text{Rot.}}{\omega_i} - i\underset{\text{Boosts}}{v_i})\right\} \qquad (4.24)$$

right-handed $$\Lambda_R = \exp\left\{ \tfrac{i}{2}\sigma^i(\omega_i + i v_i)\right\}$$

Parity

$\Lambda_L, \Lambda_R \in SL(2,\mathbb{C})$

Parity: $(x_0, \vec{x}) \xrightarrow{P} (x_0, -\vec{x})$

$$\Rightarrow \vec{J} \xrightarrow{P} \vec{J} \qquad \vec{J} \text{ pseudo-vector}$$
$$\vec{K} \xrightarrow{P} -\vec{K} \qquad \vec{K} \text{ vector} \qquad (4.25)$$

How do the $\Lambda_{L/R}$ act on coordinates:

We define $\hat{x} = x_\mu \sigma^\mu$ with $(\sigma^\mu) = (\sigma^0, \underbrace{\sigma^1, \sigma^2, \sigma^3}_{\vec{\sigma}})$

$$\sigma^0 = \mathbb{1}_2 \tag{4.26}$$

Then $\hat{x} = \begin{pmatrix} x_0 - x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 + x_3 \end{pmatrix}$

and $\det \hat{x} = x_\mu x^\mu$

Lorentz-trafo:

$$\hat{x}' = \Lambda_L \hat{x} \Lambda_L^+ \tag{4.27}$$

with $\det \hat{x}' \underset{\substack{\uparrow \\ \det \Lambda_L^{(+)} = 1}}{=} \det \hat{x}$

Remarks:

(i) Λ_L and $-\Lambda_L$ give the same $\hat{x}'$ (double covering)

(ii) $\Lambda_{L/R}^+ = \Lambda_{R/L}^{-1}$

σ maps $L \to R$

(iii) σ^μ transforms as vector

Field equations: 2-component Spinor (Weyl-spinor)

$$\mathcal{D}_L \psi_L = 0 \qquad (4.26)$$

Lorentz-trafo: $\psi_L(x) \to \Lambda_L \psi_L(x)$

$$\begin{aligned} \mathcal{D}_L \psi_L(x) &\to \mathcal{D}'_L \Lambda_L \psi_L(x) \\ &= \Lambda_R \mathcal{D} \psi_L(x) \end{aligned} \qquad (4.27)$$

$$\Rightarrow \mathcal{D}'_L = \Lambda_R \mathcal{D}_L \underbrace{\Lambda_R^+}_{= \Lambda_L^{-1}}$$

with $\mathcal{D}_L = i\bar{\sigma}^\mu \partial_\mu$, $\bar{\sigma} = (\sigma^0, -\vec{\sigma})$.

Weyl equations:

$$\boxed{\begin{aligned} i\bar{\sigma}^\mu \partial_\mu \psi_L &= 0 \\ \underbrace{i\sigma^\mu \partial_\mu}_{\mathcal{D}_R} \psi_R &= 0 \end{aligned}} \qquad (4.28)$$

Equations of motion of two-comp. spinors, no Parity invariance.

Relation to Klein-Gordon:

$$\sigma^\mu \partial_\mu \left(\bar{\sigma}^\nu \partial_\nu \psi_L = 0 \right) \tag{4.29}$$

$$= \frac{1}{2} \underbrace{\{\sigma^\mu, \bar{\sigma}^\nu\}}_{2\eta^{\mu\nu}} \partial_\mu \partial_\nu \psi_L$$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu \psi_L = 0} \tag{4.30}$$

and similarily, $\partial^2 \psi_L = 0$.

If we demand Parity invariance, we have to combine left- and right-handed spinors: Dirac spinor

$$\psi_D = \begin{pmatrix} \psi_L \\ \hline \psi_R \end{pmatrix} \begin{matrix} \text{space of left-handed ferms} \\ \text{space of right-handed ferms} \end{matrix} \tag{4.31}$$

D_L maps left- to right-handed spinors,

D_R " right- to left-handed spinors

Combination of Weyl-operators $D_{L/R}$:

$$\begin{pmatrix} 0 & \mathcal{D}_R \\ \mathcal{D}_L & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = i\gamma^\mu \partial_\mu \psi_D \qquad (4.32)$$

with [chiral rep.] $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ (4.33)

γ satisfy Clifford algebra:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}} \qquad (4.34)$$

Lorentz trafo of $i\gamma^\mu\partial_\mu\psi_D$:

4-dim Spin 1/2 representation of Λ: $\Lambda_{1/2}$

$$\Lambda_{1/2} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \qquad (4.35)$$

with $\psi_D \to \Lambda_{1/2}\psi_D = \begin{pmatrix} \Lambda_L \psi_L \\ \Lambda_R \psi_R \end{pmatrix}$

and $i\gamma^\mu\partial_\mu \psi_D \to \Lambda_{1/2}\, i\gamma^\mu \partial_\mu \Lambda_{1/2}^{-1} \Lambda_{1/2}\psi_D$

$$= \Lambda_{1/2}\, i\gamma^\mu\partial_\mu \psi_D \qquad (4.36)$$

$\Rightarrow$ Dirac equation: $\Psi = \Psi_D$

$$\boxed{(i\partial\!\!\!/ - m)\,\Psi = 0} \qquad (4.37)$$

with $\not{w} := \gamma^\mu w_\mu$.

[Lorentz-trafo: $(i\partial\!\!\!/ - m)\Psi \to \Lambda_{1/2}(i\partial\!\!\!/ - m)\Psi$]

(1) Generators $\mathcal{M}$ in spin-representation:

$$S^{\mu\nu} = i/4\,[\gamma^\mu, \gamma^\nu] \qquad (4.38)$$

with

$$[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_L & 0 \\ 0 & \left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)_R \end{pmatrix} \qquad (4.39)$$

$\sigma\,\bar\sigma : L \to L$

$\bar\sigma\,\sigma : R \to R$

Note that (see eq. (4.20), p. 101)

$$K_{i_L} = S_{0i_L} = -i\sigma_i/2 \;{}^{= i\bar\sigma_i/2} \qquad (4.40a)$$

$$J_{i_L} = \frac{1}{2}\varepsilon_{ijk} S_{jk} = -\frac{i}{2}\varepsilon_{ijk}\overbrace{[\sigma_j/2, \sigma_k/2]}^{i\varepsilon_{jkl}\sigma_l/2} = \sigma_i/2$$

Analogously:

$$K_{i_R} = i\sigma_i/2 \qquad (4.40b)$$
$$J_{i_R} = \sigma_i/2$$

and hence

$$\Lambda_{1/2} = e^{i\omega_{\mu\nu}/2 \, S^{\mu\nu}}$$
$$= \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \qquad (4.41)$$

with $\Lambda_L = \exp\left\{ i\frac{\sigma_i}{2}(\omega_i - iv_i) \right\}$

see (4.24), p. 103 $\Lambda_R = \exp\left\{ i\frac{\sigma_i}{2}(\omega_i + iv_i) \right\}$

and $\omega_{0i} = v_i$, $\omega_{ij} = \varepsilon_{ijk}\omega_k$

What is the inverse of $\Lambda_{1/2}$:

γ^0 hermitian: $\gamma^{0\,2} = \mathbb{1}$

γ^i anti-hermitian: $\gamma^{i\,2} = -\mathbb{1}$

From eq. (4.33) it also follows:

$$\gamma^0 \gamma^{i+} \gamma^0 = \gamma^i \qquad (4.42)$$

and we conclude:

$$\gamma^0 S^{\mu\nu\dagger} \gamma^0 = - S^{\mu\nu} \qquad (4.43)$$

$$\Rightarrow \boxed{\gamma^0 \Lambda_{1/2}^{\dagger} \gamma^0 = \Lambda_{1/2}^{-1}} \qquad (4.44)$$

(2) Klein - Gordon equation:

$$(-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\psi$$
$$= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\,\psi$$
$$= (\tfrac{1}{2}\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2\eta^{\mu\nu}} \partial_\mu \partial_\nu + m^2)\,\psi$$
$$\Rightarrow \boxed{(\partial_\mu \partial^\mu + m^2)\,\psi = 0} \qquad (4.45)$$

(3) Lagrangean: Lorentz scalar $\sim (i\not{\partial} - m)\psi$

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi$$
$$\xrightarrow{\Lambda} \overline{\psi}'\Lambda_{1/2}(i\not{\partial} - m)\psi \qquad (4.46)$$
$$\Rightarrow \boxed{\overline{\psi}' = \overline{\psi}\,\Lambda_{1/2}^{-1}} \qquad (4.47)$$

It follows that

$$\overline{\psi} = \psi^\dagger \gamma^0 \qquad \text{Dirac conjugate}$$

$$\text{with } \overline{\psi}' = \psi^\dagger \Lambda_{1/2}^\dagger \gamma^0 = \psi^\dagger \gamma^0 \gamma^0 \Lambda_{1/2}^\dagger \gamma^0 \qquad (4.48)$$

$$\text{eq. } (4.44) \rightarrow \; = \overline{\psi}\, \Lambda_{1/2}^{-1}$$

EoM:

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 = (i\not{\partial} - m)\psi \qquad (4.49)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0 = \overline{\psi}(i\overleftarrow{\not{\partial}} - m) = 0$$

$$\text{with } \quad f \overleftarrow{\partial_\mu} = -\partial_\mu f$$

Hamiltonian:

$$\mathcal{H} = \underset{i\overline{\psi}\gamma^0}{\Pi_\psi}\, \dot{\psi} - \mathcal{L} \qquad (4.50)$$

$$= \psi^\dagger \gamma^0 (i\underbrace{\vec{\gamma}\,\vec{\partial}}_{\gamma^i \partial_i \,=\, \gamma^i \frac{\partial}{\partial x^i}} + m)\psi$$

(4) Invariants & general properties

The derivations above made use of a specific representation of our spinors in left- and right-handed Weyl spinors. In particular for massive Dirac fermions this is not the best-adapted representation. The γ's, ψ's can be rotated with unitary transformations $\mathcal{U}$, without changing the Lagrangian in eq. (4.67)

$$\gamma \to \mathcal{U}^{+} \gamma \,\mathcal{U} \qquad (4.51)$$

(i) Clifford algebra unchanged under (4.51)

(ii) Generator $S_{\mu\nu} \to \mathcal{U}^{+} S_{\mu\nu} \mathcal{U}$

How do we project on left/right-handed eigenspaces?

Define: $$\boxed{\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3} \qquad (4.52)$$

with

(a) $\gamma_5^2 \overset{\text{Clifford}}{=} \mathbb{1}$ $\longrightarrow$ eigen values ± 1

(b) $\{\gamma_5, \gamma^\mu\} = 0$

(c) $[S_{\mu\nu}, \gamma_5] = 0$ $\rightarrow$ $S_{\mu\nu}, \gamma_5$ can be diagonalised at the same time

It follows $$\boxed{P_{L/R} = \frac{\mathbb{1} \mp \gamma_5}{2}} \qquad (4.53)$$

$P_{L/R}$ projection operators on L/R-spaces with $P_{L/R}^2 = P_{L/R}$ and $P_L + P_R = \mathbb{1}$:

$$P_{L/R}\psi = \psi_{L/R} \qquad (4.54)$$

(iii) Invariants, vectors, tensors from spinor bilinears:

Have already: $\bar{\Psi}\Psi$ scalar

$\bar{\Psi}\gamma^\mu\Psi$ vector

We use that $\gamma^\mu \to \Lambda^\mu{}_\nu \gamma^\nu$:

$\bar{\Psi} T \Psi$:

	T		
	$\mathbb{1}$	scalar	1
	γ^μ	vector	4
$\gamma^{[\mu}\gamma^{\nu]} :=$	$[\gamma^\mu, \gamma^\nu]$	tensor	6
	$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$	pseudo-vector	4
	γ_5	pseudo-scalar	1
		spans 4x4 matrices $\in$	16

Current conservation: $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, $j_5^\mu = \bar{\Psi}\gamma_5\gamma_\mu\Psi$

(a) $$\partial_\mu \underbrace{\bar{\Psi}\gamma^\mu\Psi}_{\text{current } j^\mu} \underset{\text{EoM}}{=} im\bar{\Psi}\Psi - im\bar{\Psi}\Psi = 0 \qquad (4.55)$$

Symmetry: $\Psi \to e^{i\alpha}\Psi \Rightarrow \bar{\Psi} \to \bar{\Psi}e^{-i\alpha}$

(b) $\partial_\mu \underbrace{\bar{\Psi} \gamma^\mu \gamma_5 \Psi}_{\text{axial current } j_5^\mu} = 2im \bar{\Psi} \gamma_5 \Psi$ (4.56)

conserved for $m=0$ (chiral symmetry)

Symmetry: $\Psi \to e^{i\gamma_5 \alpha} \Psi \Rightarrow \bar{\Psi} \to \bar{\Psi} e^{i\gamma_5 \alpha}$

(iv) Solutions of the Dirac equation:

As $\Psi(x)$ satisfies KG-eq.: $(\partial_\nu \partial^\nu + m^2)\Psi = 0$,

we write

$$\Psi(x) = u(p)\, e^{-ipx} \qquad (4.57)$$

with $p^2 = m^2$

$$\Rightarrow e^{ipx}(i\not{\partial} - m)\Psi(x) = (\not{p} - m)\, u(p) = 0 \qquad (4.58)$$

Similarly, with $\Psi(x) = v(p)\, e^{ipx}$

$$(\not{p} + m)\, v(p) = 0 \qquad (4.59)$$

with $p^2 = m^2$

In rest frame: $p = (p_0, 0)$

$$(\gamma^0 - \mathbb{1})\, u(p) = 0 \tag{4.60}$$

chiral rep.: $(\gamma^0 - \mathbb{1}) = \begin{pmatrix} -\mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix}$

see eq. (4.33), p. 107

Dirac rep.: $\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ (4.61)

$$\Rightarrow (\gamma^0 - \mathbb{1}) = 2\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

$$\Rightarrow \boxed{u_s(p_0^0) = \sqrt{2m}\begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \quad \text{with } s = 1/2, -1/2, \quad \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \tag{4.62}$$

The general solution follows as

$$\boxed{\begin{aligned} u_s(p) &= \frac{1}{\sqrt{2m}} \frac{\not{p} + m}{\sqrt{p^0 + m}}\, u_s(p_0^0) \\ &= \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0 + m}\chi_s \end{pmatrix} \end{aligned}} \tag{4.63}$$

Similarily, $v_s(p) = \frac{1}{\sqrt{2m}} \frac{\not{p} - m}{\sqrt{p^0 + m}} v_s(p^0)$, $v_s(p^0) = \sqrt{2m} \begin{pmatrix} 0 \\ \varepsilon \chi_s \end{pmatrix}$

$$v_s(p) = -\sqrt{p^0 + m} \begin{pmatrix} \frac{\vec{\bar{\sigma}} \cdot \vec{p}}{p^0 + m} \varepsilon \chi_s \\ \varepsilon \chi_s \end{pmatrix} \qquad (4.64)$$

with $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ← metric in spinor space

Note that

$$\varepsilon^{-1} \sigma \varepsilon = \bar{\sigma} \qquad (4.65)$$

Normalisation & relations:

$$\begin{aligned} \bar{u}_r(p)\, u_s(p) &= 2m\, \delta_{rs} \\ \bar{v}_r(p)\, v_s(p) &= -2m\, \delta_{rs} \qquad (4.66) \\ \bar{u}_r(p)\, v_s(p) &= 0 = \bar{v}_r(p)\, u_s(p) \end{aligned}$$

$$\begin{aligned} \sum_s u_s(p)_\xi\, \bar{u}_s(p)_{\bar{\xi}} &= (\not{p} + m)_{\xi\bar{\xi}} \\ &\qquad\qquad (4.67) \\ \sum_s v_s(p)_\xi\, \bar{v}_s(p)_{\bar{\xi}} &= (\not{p} - m)_{\xi\bar{\xi}} \end{aligned}$$

Eq. (467) is proven by showing it on the basis $u_s(p)$, $v_s(p)$.

(4.66) :

(i) $\bar{u}_r(p)\, u_s(p) = u_r^\dagger(p^0)\, \frac{(\not{p}^\dagger + m)\gamma^0(\not{p}+m)}{p^0+m}\, u_s(p^0)$

$\gamma^0\gamma^\dagger\gamma^0 = \gamma \Rightarrow \qquad = u_r^\dagger(p^0)\,\gamma^0\, \frac{(\not{p}+m)(\not{p}+m)}{p^0+m}\, u_s(p^0)$

$u_r^\dagger(p^0) = u_r(p^0) \qquad = u_r(p^0)\,\gamma^0\, \frac{(p^2+m^2+2p^0 m\gamma^0)}{p^0+m}\, u_s(p^0)$

$u_r(p^0)\gamma^0\gamma^i u_s(p^0) = 0$

$\gamma^0 u_s(p^0) = u_s(p^0) \qquad = 2u_r(p^0)\,\gamma^0\left(\frac{m(\cancel{p^0+m})}{\cancel{p^0+m}}\right) u_s(p^0)$

$= 2m\,\delta_{rs}$

(ii) $\bar{v}_r v_s$ follows as in (i)

(iii) $\bar{u}_r(p)\, v_s(p) = 0$ from $(\not{p}-m)(\not{p}+m) = 0$

(4.67)

$$\sum_s u_s(p) \underbrace{\bar{u}_s(p)\cdot u_r(p)}_{2m\delta_{rs}}$$

$$= 2m\, u_r(p) = \frac{2m(\not{p}+m)}{\sqrt{p^0+m}}\, u_r(p^0) = \frac{(\not{p}+m)^2}{\sqrt{p^0+m}}\, u_r(p^0)$$

$$= (\not{p}+m)\, u_r(p)$$

$$\sum_s u_s(p)\ \bar{u}_s(p)\, v_r(p) = 0$$

$$= (\not{p}+m)\, V_r(p)$$

Similarily for $\sum_s v_s(p)\ \bar{v}_s(p)$