4.3 Quantisation

First we try to quantise fermions similarily to scalars (bosons).
The general solution to the Dirac-equation follows as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_s \left\{ e^{-ipx} a_s(\vec{p}) \cdot u_s(\vec{p}) + e^{ipx} b_s^+(\vec{p}) v_s(\vec{p}) \right\} \quad (4.68)$$

Question: What are the properties of the creation/annihilation operators a, b / a^+, b^+ ?

(i) Hamiltonian (see eq. (4.50), p. 112)

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \underbrace{\psi^\dagger(\vec{x})}_{-i\Pi_\psi} \gamma^0 (i\vec{\gamma}\vec{\partial} + m)\psi(\vec{x})$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0}\, p_0 \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s(\vec{p})\, b_s^\dagger(\vec{p}) \right\} \qquad (4.69)$$

where we have used (4.66) and

$$(\vec{\gamma}\,\vec{p} + m)\, u(p) = \left[-(\not{p} - m) + \gamma^0 p^0\right] u(p)$$

$$= \gamma^0 p^0 u(p) \qquad (4.70)$$

and $(-\vec{\gamma}\,\vec{p} + m)\, v(p) = -\gamma^0 p^0 v(p)$.

minus in (4.69)

Using commuting operators, e.g.

$$b_s b_s^\dagger = b_s^\dagger b_s + c\text{-number}$$

$$\Rightarrow H \simeq \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, p_0 \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\} \qquad (4.71)$$

⇒ suggests the use of $b_s b_s^\dagger = -b_s^\dagger b_s + c\text{-number}$

(iii) Demanding

$$[\psi(\vec{x}), i\psi^\dagger(\vec{y})] = i\, \delta^{(3)}(\vec{x}-\vec{y}) \qquad (4.72)$$

implies

$$[a_s(\vec{p}), a_r^\dagger(\vec{q})] = 2p^0 \delta^3(\vec{p}-\vec{q})$$

$$= -[b_s(\vec{p}), b_r^\dagger(\vec{q})]$$

! rescues causality

but does not cure (4.71) !

We conclude

$$\boxed{\begin{aligned} \{a_s(\vec{p}), a_r^\dagger(\vec{q})\} &= (2\pi)^3 2p^0 \delta_{sr} \delta^3(\vec{p}-\vec{q}) \\ \{b_s(\vec{p}), b_r^\dagger(\vec{q})\} &= (2\pi)^3 2p^0 \delta_{sr} \delta^3(\vec{p}-\vec{q}) \end{aligned}} \qquad (4.73)$$

b-b, a-a, b-$a^{(\dagger)}$ anticommutators vanish,

in particular $a_s(\vec{p})\, a_s(\vec{p}) = 0$

a Grassmann variable

and hence

$$\boxed{\begin{aligned} &\{ \psi_\xi(\vec{x}), \psi^\dagger_{\xi'}(\vec{y}) \} = \delta_{\xi\xi'} \delta^3(\vec{x}-\vec{y}) \\ &\{ \psi_\xi(\vec{x}), \psi_\xi(\vec{y}) \} = 0 = \{ \psi^\dagger_\xi(\vec{x}), \psi^\dagger_\xi(\vec{y}) \} \end{aligned}} \qquad (4.74)$$

(iii) Fock space

Vacuum $|0\rangle$: $a_s(\vec{p})|0\rangle = b_s(\vec{p})|0\rangle = 0$

1 Particle states: $a_s^\dagger(\vec{p})|0\rangle = |\vec{p}, s\rangle$

anti-particle: $b_s^\dagger(\vec{p})|0\rangle$ (4.75)

2 particle states: $a^\dagger_s(\vec{p})\, a^\dagger_r(\vec{q})|0\rangle$

$$= \underset{\uparrow\,!}{-}\, a^\dagger_r(\vec{q})\, a^\dagger_s(\vec{p})|0\rangle \qquad (4.76)$$

$$\boxed{\text{States are anti-symmetric}}$$

in particular:

$$\boxed{a^\dagger_r(\vec{p})\, a^\dagger_r(\vec{p})|0\rangle = 0} \qquad (4.77)$$

Normalisation: $\langle \vec{q}, r | \vec{p}, s\rangle = (2\pi)^3\, 2p^0\, \delta^3(\vec{p}-\vec{q})\, \delta_{rs}$

(iv) Cont. symmetries

Four-momentum:

$$P^0 = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\} \tag{4.78a}$$

$= H$ Fermions, Anti-fermions with $E = p^0 > 0$

$$P^i = \int \frac{d^3p}{(2\pi)^3} \frac{p^i}{2p^0} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\} \tag{4.78b}$$

Ψ is a complex field, the Lagrangian is invariant under $\Psi \to e^{ie\alpha}\Psi$, $\bar{\Psi} \to \bar{\Psi} e^{-ie\alpha}$

see p. 116, eq. (4.56)

Noether charge: $Q = \int d^3x\, j^0 = e\int d^3x\, \Psi^\dagger(x)\Psi(x)$

$$\Rightarrow \boxed{Q \simeq e\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\}} \tag{4.79}$$

elementary charge; Fermion with charge e; Anti-Fermion with charge $-e$

(v) Propagator:

$$\langle 0|\, \psi_\xi(x)\, \bar\psi_{\xi'}(y)\, |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_s (u_s)_\xi (\bar u_s)_{\xi'}\, e^{-ip(x-y)} \qquad (4.80a)$$

$$= (i\not\partial_x + m)_{\xi\xi'} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, e^{-ip(x-y)}}_{\text{Scalar prop.}}$$

and

$$\langle 0|\, \bar\psi_{\xi'}(y)\psi_\xi(x)\, |0\rangle$$

$$= -(i\not\partial_x + m)_{\xi\xi'} \overbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, e^{-ip(y-x)}} \qquad (4.80b)$$

global minus sign

$\Rightarrow$ Feynman propagator:

$$S_F(x-y) = \langle 0|\, T\, \psi_\xi(x)\, \bar\psi_{\xi'}(y)\, |0\rangle \qquad (4.81)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x-y)}$$

with time-ordering

$$T\,\psi(x)\bar\psi(y) = \Theta(x^0-y^0)\,\psi(x)\,\bar\psi(y) - \Theta(y^0-x^0)\,\bar\psi(y)\,\psi(x) \qquad (4.82)$$

Feynman rules:

We can directly take over the results for the scalar theory, chapter 3.2 – 3.5, but we have to take care of the anti-symmetry of fermions.

We have already introduced

$$T\psi\bar{\psi} = -T\bar{\psi}\psi$$

Accordingly, if we define contractions as in the scalar theory, it follows

$$\begin{aligned} \psi(x)\bar{\psi}(y) &= \langle 0 | T\, \psi(x)\, \bar{\psi}(y) | 0 \rangle \\ &= S_F(x-y) \qquad\qquad (4.83) \\ &= -\bar{\psi}(y)\,\psi(x) \end{aligned}$$

Furthermore

$$\ldots \psi\ \psi^n\ \bar{\psi}^m\ \bar{\psi} \ldots = (-1)^{n+m} \ldots \psi\bar{\psi}\ \psi^n\ \bar{\psi}^m$$

(4.84)

Also $\quad :a\,a^\dagger: = -:a^\dagger a: = a^\dagger a \qquad (4.85)$

$$\Rightarrow :\psi_1 \cdots \psi_n \overset{(-)}{\psi}_{n+1} \cdots :$$

$$= -:\psi_1 \cdots \overset{(-)}{\psi}_{n+1} \psi_n \cdots : \qquad (4.86)$$

$\Rightarrow$ Wick's theorem: (eq. (3.52))

$$\boxed{\begin{aligned} &T\, \psi(x_1) \cdots \psi(x_n) \bar\psi(x_{n+1}) \cdots \bar\psi(x_{n+m}) \\ &\quad = :\psi(x_1) \cdots \bar\psi(x_{n+m}) + \text{all contractions}: \qquad (4.87)\end{aligned}}$$

Simpl(est) interacting theory:

$$\mathcal{L} = \mathcal{L}_{scalar} + \mathcal{L}_{Dirac} + \mathcal{L}_I \qquad (4.88)$$

with

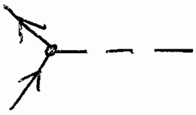
$$\mathcal{L}_I = -h\,\bar\psi \phi \psi \qquad (4.89)$$

Yukawa theory

Propagators:

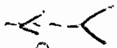
(a) ϕ: $\quad \phi\,\phi \doteq$  p $= \dfrac{i}{p^2 - m_\phi^2 + i\varepsilon} \qquad (4.90)$

(b) ψ: $\quad \psi\,\bar\psi =$  p $= \dfrac{i(\not p + m_\psi)}{p^2 - m_\psi^2 + i\varepsilon} \qquad (4.91)$

Vertex:  $= -ig$

External leg contraction

(a) $\quad \phi\,|\vec p\rangle := 1 =: \langle \vec p\,|\,\phi$

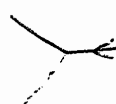
 p  p

(b)

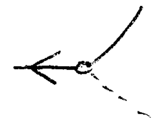
$$\psi(x)\Big|_{\text{annihil.}} |\vec p, s\rangle = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\sum_r e^{-iqx}\, a_r(\vec q)\, a_s^\dagger(\vec p)|0\rangle$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\sum_r e^{-iqx}\left\{a_r(\vec q), a_s^\dagger(\vec p)\right\}|0\rangle$$

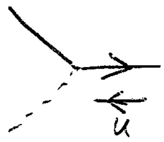
$$= e^{-ipx}\, u_s(\vec p) \qquad (4.92)$$

$$\Rightarrow \boxed{\psi\,|\vec p, s\rangle := u_s(\vec p)} =$$  $p \qquad (4.93)$

Analogously:

$$\langle \vec{p}, s | \bar{\psi} := \bar{u}_s(\vec{p}) = $$  (4.94)

Anti-fermions:

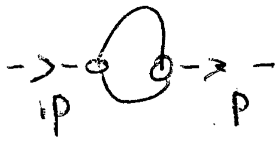


$$= \bar{\psi} | \vec{k}, s \rangle := \bar{v}^s(\vec{k}) \qquad (4.95)$$

$$\qquad \| $$

$$b_s^+(\vec{k}) | 0 \rangle$$

$$\langle \vec{k}, s | \psi := v^s(\vec{k}) \qquad (4.96)$$

Loops: e.g. 

$$\Rightarrow \langle \vec{q} | (\phi \, \bar{\psi} \, \psi)(\phi \, \bar{\psi} \, \psi) | \vec{p} \rangle \qquad (4.97)$$

$$\sim - \langle \vec{q} | \phi \, (\psi \bar{\psi} \; \psi \bar{\psi}) \, \phi | \vec{p} \rangle$$

closed fermionic loops lead to minus signs

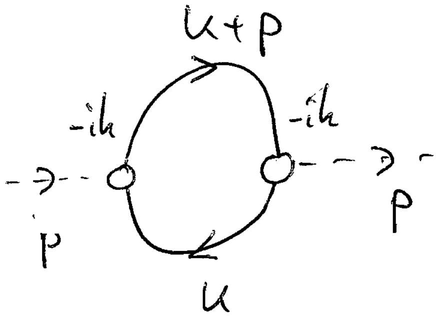
Loop integral & Dirac traces on p. 128a

$$\Rightarrow \quad -(-ih)^2 \int d^4x \int d^4y \, \langle \vec{q} \,|\, \phi_x \, \psi_{x\,\xi} \, \bar{\psi}_{y\,\eta} \, \psi_{y\,\eta} \, \bar{\psi}_{x\,\xi} \, \phi_y \,|\, \vec{p} \rangle$$

trace

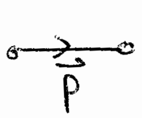
$$\simeq \; - h^2 \int \frac{d^4 k}{(2\pi)^4} \, tr\left(\frac{\not{k} + m_\psi}{p^2 - m_\psi^2} \; \frac{\not{k} + \not{p} + m_\psi}{(k+p)^2 - m^2} \right)$$

from



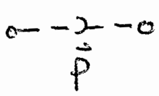
In summary: (see p. 68)

Feynman rules in momentum space

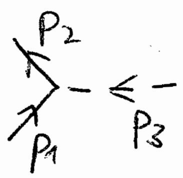
(i) 

$= \dfrac{\not{p} + m_\psi}{p^2 - m_\psi^2 + i\varepsilon}$

(4.98)



$= \dfrac{1}{p^2 - m_\phi^2 + i\varepsilon}$

(ii) 

$= -ih$ and $p_1 + p_3 = p_2$ (4.9

(iii) $\int \frac{d^4p}{(2\pi)^4}$ for each loop

(−) for each fermion loop

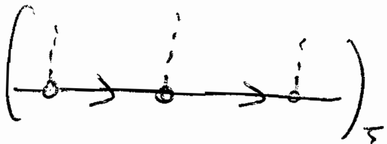
(iv) $(2\pi)^4 \delta^4(\sum_i p_i)$ for 

Remarks: (a) no symmetry factor!

$\mathcal{L}_I$ is built-up from 3 diff. fields

(b) momentum direction of fermion line <u>important</u>!

(c) Dirac indices are contracted along fermion lines, e.g.

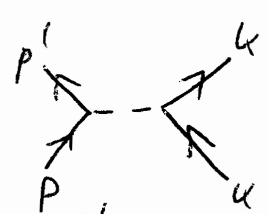


$$\left(\quad \right)_{\xi\xi'} \simeq (-ih)^2 \left(\frac{\not p + m}{p^2-m^2+i\varepsilon}\right)_{\xi\eta} (-ih) \left(\frac{\not p + m}{p^2-m^2+i\varepsilon}\right)_{\eta\xi'} \tag{4.100}$$

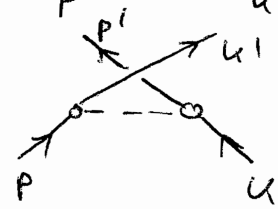
$$/\!/$$

$$\ldots\ \bar\psi\,\phi\Big[(\psi\bar\psi)\,\phi\,(\psi\bar\psi)\Big]_{\xi\xi'}\phi\,\psi\ \ldots$$

Exercise: scattering process



$+$



$$\Rightarrow i\mathcal{M} = (-ih)^2 \Big[\bar u(\vec p\,') u(\vec p) \frac{1}{(p-p')^2 - m_\phi^2} \bar u(\vec k') u(\vec k)$$

$$- \bar u(\vec p\,') u(\vec k) \frac{1}{(p-k)^2 - m_\phi^2} \bar u(\vec k') u(\vec p) \Big] \tag{4.101}$$

QED: couple electron ψ_e to photon A_μ:

$$\mathcal{L}_I = -e\,\bar{\Psi}\,A_\mu \gamma^\mu \psi$$

and

$$\mathcal{L}_{QED} = \mathcal{L}_{photon} + \mathcal{L}_{Dirac} + \mathcal{L}_I$$

with

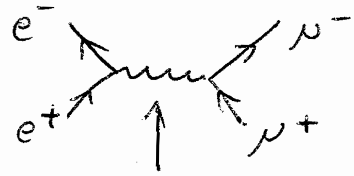
$$\mathcal{L}_{Dirac} + \mathcal{L}_I = \bar{\Psi}(i\not{D} - m)\psi$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

Vertex:  $= -ie$

Example:



photon propagator ⇒ Quantis. of gauge field

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Also $\quad :a\,a^\dagger: \;=\; -:a^\dagger a: \;=\; a^\dagger a \qquad (4.85)$

$$\Rightarrow\ :\psi_1 \cdots \psi_n \overset{(-)}{\psi}_{n+1} \cdots :$$

$$= -:\psi_1 \cdots \overset{(-)}{\psi}_{n+1}\, \psi_n \cdots : \qquad (4.86)$$

$\Rightarrow$ Wick's theorem: (eq. (3.52))

$$\boxed{\begin{aligned} &T\ \psi(x_1) \cdots \psi(x_n)\, \bar\psi(x_{n+1}) \cdots \bar\psi(x_{n+m}) \\ &\quad = :\psi(x_1) \cdots \bar\psi(x_{n+m}) + \text{all contractions}: \qquad (4.87) \end{aligned}}$$

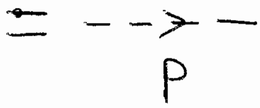
Simpl(est) interacting theory:

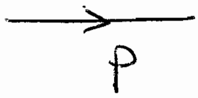
$$\mathcal{L} = \mathcal{L}_{scalar} + \mathcal{L}_{Dirac} + \mathcal{L}_I \qquad (4.88)$$

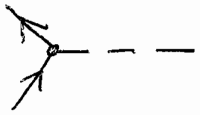
with $\quad \mathcal{L}_I = -h\, \bar\psi \phi \psi \qquad (4.89)$

Yukawa theory

Propagators:

(a) ϕ:
$$\phi\,\phi \doteq$$

$$= \frac{i}{p^2 - m_\phi^2 + i\varepsilon} \qquad (4.90)$$

(b) ψ:
$$\psi\,\bar\psi =$$

$$= \frac{i(\not p + m_\psi)}{p^2 - m_\psi^2 + i\varepsilon} \qquad (4.91)$$

Vertex:

$$= -ig$$

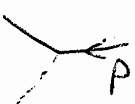
External leg contraction

(a) $$\phi\,|\vec p\rangle := 1 =: \langle \vec p\,|\,\phi$$


(b) $$\underbrace{\psi(x)|}_{\text{annihil.}}\,|\vec p, s\rangle = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\sum_r e^{-iqx}\, a_r(\vec q)\, a_s^\dagger(\vec p)|0\rangle$$

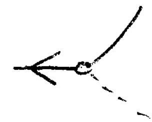
$$= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\sum_r e^{-iqx}\left\{a_r(\vec q), a_s^\dagger(\vec p)\right\}|0\rangle$$

$$= e^{-ipx}\, u_s(\vec p) \qquad (4.92)$$

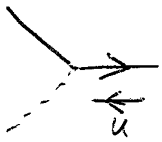
$$\Rightarrow \boxed{\psi\,|\vec p, s\rangle := u_s(\vec p)} =$$

$$(4.93)$$

Analogously:

$$< \vec{p}, s | \bar{\Psi} := \bar{u}_s(\vec{p}) = \quad (4.94)$$



Anti-fermions:



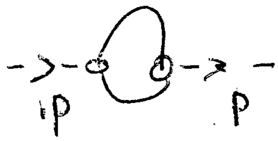
$$= \bar{\Psi} | \vec{k}, s > := \bar{v}^s(\vec{k}) \quad (4.95)$$

$$\| $$

$$b_s^+(\vec{k}) | 0 >$$

$$< \vec{k}, s | \psi := v^s(\vec{k}) \quad (4.96)$$

Loops: e.g.



$$\Rightarrow < \vec{q} | (\phi \bar{\Psi} \psi)(\phi \bar{\Psi} \psi) | \vec{p} > \quad (4.97)$$

$$\sim - < \vec{q} | \phi (\psi \bar{\Psi} \; \psi \bar{\Psi}) \phi | \vec{p})$$

closed fermionic loops lead to minus signs

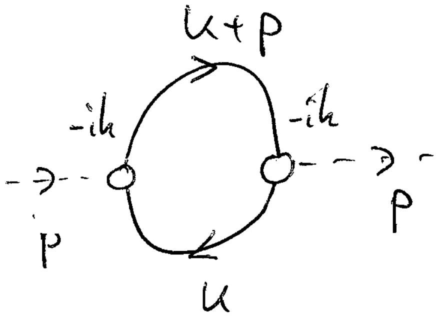
Loop integral & Dirac traces on p. 128a

$$\Rightarrow \quad -(-ih)^2 \int d^4x \int d^4y \, \langle \vec{q} \,|\, \phi_x \, \psi_{x\,\xi} \, \bar{\psi}_{y\,\eta} \, \psi_{y\,\eta} \, \bar{\psi}_{x\,\xi} \, \phi_y \,|\, \vec{p} \rangle$$

trace

$$\simeq \; -h^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr}\left(\frac{\not{k} + m_\psi}{p^2 - m_\psi^2} \; \frac{\not{k} + \not{p} + m_\psi}{(k+p)^2 - m^2} \right)$$

from



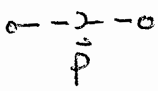
In summary: (see p. 68)

Feynman rules in momentum space

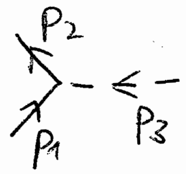
(i) 

$$= \frac{\not{p} + m_\psi}{p^2 - m_\psi^2 + i\varepsilon}$$

(4.98)



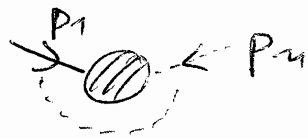
$$= \frac{1}{p^2 - m_\phi^2 + i\varepsilon}$$

(ii) 

$= -ih$ and $p_1 + p_3 = p_2$ (4.9

(iii) $\int \frac{d^4p}{(2\pi)^4}$ for each loop

(−) for each fermion loop

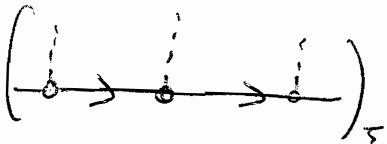
(iv) $(2\pi)^4 \, \delta^4(\sum_i p_i)$ for 

Remarks: (a) no symmetry factor!

$\mathcal{L}_I$ is built-up from 3 diff. fields

(b) momentum direction of fermion line <u>important</u>!

(c) Dirac indices are contracted along fermion lines, e.g.

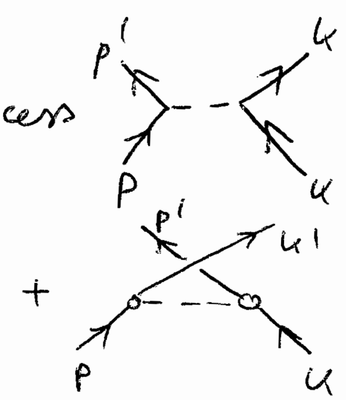


$$\Big(\;\cdots\;\Big)_{\xi\xi'} \simeq (-ih)^2 \left(\frac{\not p + m}{p^2 - m^2 + i\varepsilon}\right)_{\xi\eta} (-ih) \left(\frac{\not p + m}{p^2 - m^2 + i\varepsilon}\right)_{\eta\xi'} \tag{4.100}$$

$$/\!/$$

$$\dots \bar\psi\,\phi\left[(\psi\bar\psi)\,\phi\,(\psi\bar\psi)\right]_{\xi\xi'}\phi\,\psi \dots$$

Exercise: scattering process



$$\Rightarrow i\mathcal{M} = (-ih)^2 \Big[\bar u(\vec p\,')u(\vec p)\frac{1}{(p-p')^2 - m_\phi^2}\;\bar u(\vec k')u(\vec k)$$

$$- \bar u(\vec p\,')u(\vec k)\frac{1}{(p-k)^2 - m_\phi^2}\bar u(\vec k')u(\vec p)\Big] \tag{4.101}$$

QED: couple electron Ψ_e to photon A_μ:

$$\mathcal{L}_I = e\,\overline{\Psi}\,A_\mu \gamma^\mu \psi$$

and

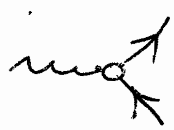
$$\mathcal{L}_{QED} = \mathcal{L}_{Photon} + \mathcal{L}_{Dirac} + \mathcal{L}_I$$

with

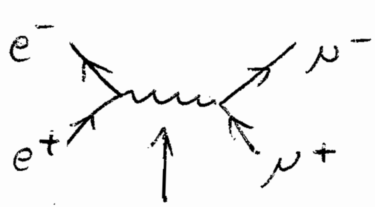
$$\mathcal{L}_{Dirac} + \mathcal{L}_I = \overline{\Psi}(i\not{D} - m)\psi$$

where

$$D_\mu = \partial_\mu - ie A_\mu$$

Vertex:  $= ie$

Example:



photon propagator $\Rightarrow$ Quantis. of gauge field