5 Gauge fields

5.1 Gauge symmetry

Consider Dirac theory of e^+, e^-

$$\mathcal{L}_D = \bar{\Psi}(x)\,(\,i\not{\partial} - m\,)\,\Psi(x) \qquad (5.1)$$

or complex scalar theory

$$\mathcal{L}_\phi = \partial_\mu \phi\, \partial_\mu \phi^* - m^2 \phi\phi^* - V(\phi\phi^*) \qquad (5.2)$$

The Lagrangians (5.1), (5.2) are invariant under global $U(1)$-rotations:

$$\begin{aligned} \Psi &\to e^{i\alpha}\Psi \;, & \bar{\Psi} &\to \bar{\Psi} e^{-i\alpha} \\ \phi &\to e^{i\alpha}\phi \;, & \phi^* &\to \phi^* e^{-i\alpha} \end{aligned} \qquad (5.3)$$

Global rotation in field space

Let us require the invariance of the theory under <u>local</u> rotations (gauge sym.)
e.g.

$$\psi(x) \to e^{i\alpha(x)} \psi(x) \qquad (5.4)$$

$\mathcal{L}_D$ is not invariant,

$$\begin{aligned} \mathcal{L}_D &\to \mathcal{L}_D - \bar{\Psi}(\not{\partial}\alpha)\Psi \\ &= \mathcal{L}_D - \partial_\nu \alpha\, j^\nu \end{aligned} \qquad (5.5)$$

with $j^\nu = \bar{\Psi}\gamma^\nu\Psi$. Hence, if we add a term $A_\nu j^\nu$ to $\mathcal{L}_D$, and demand invariance, it follows

$$\begin{aligned} \mathcal{L}_D + A_\nu j^\nu \to\ & \mathcal{L}_D - \partial_\nu\alpha\, j^\nu + A'_\nu j^\nu \\ \overset{!}{=}\ & \mathcal{L}_D + A_\nu j^\nu \end{aligned}$$

$$\Rightarrow \quad A_\nu \to A_\nu + \partial_\nu \alpha \qquad (5.6)$$

Also: $\mathcal{L}$ Lorentz scalar: $A_\nu \xrightarrow{\Lambda} \Lambda^\nu{}_\nu A^\nu$ (5.7)
as $A_\nu j^\nu$ scalar

We write the invariant action

$$\mathcal{L}_D = \overline{\Psi}(i\not{D} - m)\Psi \qquad (5.8)$$

with covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu \qquad (5.9)$$

A_μ is also called a connection, it induces covariant transformation properties for $\mathcal{D}_\mu$:

$$\mathcal{D}_\mu \to e^{i\alpha(x)}\mathcal{D}_\mu e^{-i\alpha(x)}$$

$$\Rightarrow \quad \mathcal{D}_\mu \Psi \to e^{i\alpha(x)}\mathcal{D}_\mu \Psi$$ transforms homog. as the field Ψ (5.10)

as well $\mathcal{D}_\mu \phi \to e^{i\alpha(x)}\mathcal{D}_\mu \phi$

Similarily we get

$$\mathcal{L}_\phi = \mathcal{D}_\mu \phi (\mathcal{D}_\mu \phi)^* - m^2 \phi\phi^* - V(\phi\phi^*) \qquad (5.11)$$

is invariant under $\phi(x) \to e^{i\alpha(x)}\phi(x)$

Dynamics of gauge field A_μ:

(i) Construct gauge-invariant scalar quantities from A_μ. This is easily done from D_μ, which transforms covariantly:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -i(\partial_\mu A_\nu - \partial_\nu A_\mu) = -i F_{\mu\nu} \tag{5.12}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, or $F_{\mu\nu} = i[D_\mu, D_\nu]$ (5.13)

Lorentz trafos: A_μ transforms as vector, see eq. (5.7)

$$\Rightarrow F_{\mu\nu} \to \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$$

transforms as tensor

$F_{\mu\nu}$: field strength, curvature

gauge invariant: $F_{\mu\nu} \to F_{\mu\nu}$ (5.14)

with $D_\mu \to e^{i\alpha} D_\mu e^{-i\alpha}$

In summary: $F_{\mu\nu}$ gauge inv.

$F_{\mu\nu} F^{\mu\nu}$ gauge inv. Lorentz scalar

$\Rightarrow$ Gauge-inv. Lagrangian:

$$\mathcal{L}_{QED} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_D \qquad (5.15)$$

Reparameterise $A_\mu \to e \, A_\mu$

$\uparrow$ electric charge

$$\Rightarrow \quad \mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D} - m)\psi \qquad 5.16$$

with $D_\mu = \partial_\mu - i e A_\mu$

Remark: construction also goes through for $\psi \to U \psi$ for $U \in SU(N)$

$$F_{\mu\nu} F^{\mu\nu} \to \sum_{a=1}^{N^2-1} F_{\mu\nu}{}^a F^{\mu\nu a}$$

with $F_{\mu\nu} = i/e \, [D_\mu, D_\nu] \sim [A_\mu, A_\nu]$