5.2 Quantisation

We concentrate on the pure gauge field Lagrangian:

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.17}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

EoM:

$$\partial_\mu F^{\mu\nu} = \left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right) A_\sigma = 0 \tag{5.18}$$

with current $\partial_\mu F^{\mu\nu} = J^\nu$

Eq. (5.18) reflects the redundancy of the gauge field A_μ: $A_\mu \to A_\mu + e\,\partial_\mu \alpha$

$$\left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right) \partial_\sigma \alpha(x) = 0 \tag{5.19}$$

Problems:

(i) The eqs. (5.18), (5.19) already entail that A^ν cannot have canonical commutation relations! What about the canonical momentum π^ν:

$$\pi^\nu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\nu} = -\frac{1}{4} \frac{\partial}{\partial \partial_0 A_\nu} \left(F_{\rho\sigma} F_{\gamma\delta} \eta^{\sigma\delta} \eta^{\rho\gamma} \right)$$

$$= -\frac{1}{2} F_{\rho\sigma} \eta^{\sigma\delta} \eta^{\rho\gamma} \frac{\partial F_{\gamma\delta}}{\partial \partial_0 A_\nu}$$

$$= F^{\nu 0} \qquad (5.20)$$

In particular: $\boxed{\pi^0 = 0} \Leftarrow$ reflects redundancy

(ii) Remove redundancy by fixing the gauge, e.g. Lorentz -or covariant gauge:

$$\partial_\nu A^\nu = 0 \qquad (5.21)$$

Remark: For A^μ with (5.21) we can write

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\underbrace{\partial_\mu A^\mu}_{\text{is } 0})^2 \qquad (5.22)$$

or $$S[A] = \frac{1}{2}\int d^4x \, A_\mu \left(\partial_\rho \partial^\rho \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right) A_\nu \qquad (5.23)$$

and EoM $$\partial_\mu F^{\mu\nu} = -\frac{1}{\xi}\partial^\nu (\underbrace{\partial_\mu A^\mu}_{\text{is } 0})$$

Note that $\left(\partial_\rho \partial^\rho \eta^{\mu\nu} - (1 - \frac{1}{\xi})\partial^\mu \partial^\nu\right)$ is invertible, it is specifically simple for $\boxed{\xi = 1}$.

With the gauge (5.21) (or $\xi = 1$) the EoM read

$$\boxed{\partial_\rho \partial^\rho A^\nu = 0} \qquad (5.24)$$

KG - equation

Eq. (5.24) suggest a quantised field

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \left\{ e^{-ikx} a_\mu(\vec{k}) + e^{ikx} a_\mu^\dagger(\vec{k}) \right\} \tag{5.25}$$

with commutation relations

$$\boxed{\left[a_\mu(\vec{k}), a_\nu(\vec{k}') \right] = -\boxed{\eta_{\mu\nu}} (2\pi)^3 \, 2k^0 \, \delta^3(\vec{k}-\vec{k}')} \tag{5.26}$$

necessary for Lorentz-sy

and $[a_\mu(\vec{k}), a_\nu(\vec{k}')] = 0 = [a_\mu^\dagger(\vec{k}), a_\nu^\dagger(\vec{k}')]$

However eqs. (5.25), (5.26) are not compatible with eq. (5.21):

$$\partial_\mu A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{-i}{2k^0} \left\{ e^{-ikx} k^\mu a_\mu(\vec{k}) + e^{ikx} k^\mu a_\mu^\dagger(\vec{k}) \right\}$$

$$\overset{!}{=} 0 \tag{5.27}$$

This entails that $k^\mu a_\mu(\vec{k}) \overset{!}{=} 0$. Note that if (5.27) fails, the EoM is <u>not</u> satisfied: $\boxed{\partial_\mu F^{\mu\nu} = -\partial^\nu \partial_\mu A^\mu}$

However

$$k^\mu [a_\mu(\vec{k}), a^+_\nu(\vec{k}')] = - k_\nu (2\pi)^3 k^0 \delta^3(\vec{k}-\vec{k}')$$

$$\neq 0 \qquad (5.28)$$

Indeed one can show that it is <u>not</u> possible to quantise the gauge field A_μ with can. com. relations and $\partial_\mu A^\mu = 0$, or other gauge conditions: If using A^μ in (5.25), (5.26), the gauge $\partial_\mu A^\mu$ has to be implemented on the states!

(iii) Fockspace $\mathcal{F}$: standard construction based on (5.25), (5.26)

(a) vacuum $|0\rangle$ with $\langle 0|0\rangle = 1$

(b) one-particle states : $a^{+}_{\mu}(\vec{k})|0\rangle$

with norm $\langle 0|a_{\nu}(\vec{k}')\, a^{+}_{\mu}(\vec{k})|0\rangle$

$$= -\boxed{\eta_{\mu\nu}}(2\pi)^3\, 2k^0\, \delta^3(\vec{k}-\vec{k}') \tag{5.29}$$

$\mu=\nu=i$: positive norm states

$\mu=\nu=0$: negative norm states

$\Rightarrow \tilde{\mathcal{F}}$ is not the phys. Hilbert space $\mathcal{H}$, as it does not allow for prob. interpertation.

Remarks:

(i) $\eta_{\mu\nu}\to-\eta_{\mu\nu}$ does not solve the problem of negative norm states (leave aside the wrong commutators $[A^i,\pi^i]$.

(ii) Separating the positive norm sub space of $\tilde{\mathcal{F}}$ will resolve all problems (i)-(iii)

Gupta - Bleuler quantisation:

(i) We demand EoM satisfied on phys. states,

$$\boxed{\langle \text{phys. states}' | \partial_\nu F^{\nu\nu} | \text{phys. states} \rangle \overset{!}{=} 0} \quad (5.30)$$

that is, its matrix elements vanish.

Eq. (5.30) is satisfied for

$$\boxed{k^\nu a_\nu(\vec{k}) \,|\, \text{phys. states} \rangle = 0} \quad (5.31)$$

(trivially satisfied on the vacuum)

The above suggests to rewrite A_ν in (5.25)

as

$$\boxed{A_\nu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \sum_{\lambda=0}^{3} \left\{ \alpha_\lambda(\vec{k})\, \varepsilon^\lambda_\nu(k) e^{-ikx} + \alpha^\dagger_\lambda(\vec{k})\, \varepsilon^{\lambda *}_\nu(k)\, e^{ikx} \right\}} \quad (5.32)$$

where the $\varepsilon^\lambda{}_\mu$ introduce unitary rotations from a_μ to α_λ with

$$\varepsilon^\lambda{}_\mu(k)\,\varepsilon^{\lambda'\mu *}(k) = \eta^{\lambda\lambda'} \qquad (5.33)$$
$$\varepsilon^\lambda{}_\mu(k)\,\varepsilon_{\lambda\nu}{}^*(k) = \eta_{\mu\nu}$$

and hence

$$\boxed{\alpha_\lambda(\vec{k}) = a_\mu(\vec{k})\,\varepsilon_\lambda{}^\mu(k) \qquad (5.34)}$$

We choose $k\cdot\varepsilon^0 = k^0 = k\cdot\varepsilon^3$ and $k\cdot\varepsilon^i = 0$, $i=1,2$

The ε's are called polarisation vectors.

Eq. (5.31) now reads with $\alpha_\pm = \frac{1}{\sqrt{2}}(\alpha_0 \pm \alpha_3)$

$$\boxed{\ldots\ \alpha_+\,|\text{phys. states}\rangle = 0} \qquad (5.35)$$

with $\alpha_0 + \alpha_3 \simeq k^\mu a_\mu$.

In the frame with $(k^\mu) = (k^0, 0, 0, k^0)$ we have

$$(\varepsilon^\lambda)_\mu = \delta^\lambda{}_\mu \qquad (5.36)$$

The α's have the same commutation relation as the a (unitary rotation, see also (5.33), (5.34))

It follows with

$$i=1,2: \qquad [\alpha_i(\vec{k}), \alpha_i^{+}(\vec{k}')] = (2\pi)^3\, 2k^0\, \delta^3(\vec{k}-\vec{k}')$$

$$[\alpha_+(\vec{k}), \alpha_-^{+}(\vec{k}')] = -(2\pi)^3\, 2k^0\, \delta^3(\vec{k}-\vec{k}')$$

$$[\alpha_\pm(\vec{k}), \alpha_\pm^{(+)}(\vec{k}')] = 0 = [\alpha_\pm(\vec{k}), \alpha_i^{(+)}(\vec{k}')] \tag{5.37}$$

Physical Hilbertspace $\mathcal{H}$:

(i) Physical sub-space $\mathcal{F}_{phys}$: $|\Psi\rangle \in \mathcal{F}_{phys}$

$$\Rightarrow \alpha_+ |\Psi\rangle = 0$$

It follows

$$|\Psi\rangle \in \mathcal{F}_{phys} \Rightarrow \alpha_i^{+} |\Psi\rangle \in \mathcal{F}_{phys}, \; i=1,2$$

$$\text{with } \alpha_+ \alpha_i^{+} |\Psi\rangle = \alpha_i^{+} \alpha_+ |\Psi\rangle = 0 \tag{5.38}$$

Also $\alpha_+^{+} |\Psi\rangle \in \mathcal{F}_{phys}$

$$\text{with } \alpha_+ \alpha_+^{+} |\Psi\rangle = \underbrace{\alpha_+^{+} \alpha_+ |\Psi\rangle}_{0} \tag{5.39}$$

Finally $\alpha_-^+|\Psi\rangle \notin \mathcal{F}_{phys}$

since $\alpha_+ \alpha_-^+ |\Psi\rangle = \underbrace{\alpha_-^+ \alpha_+ |\Psi\rangle}_{0} + \underbrace{[\alpha_+, \alpha_-^+]|\Psi\rangle}_{\nu|\Psi\rangle \neq 0}$ (5.40)

We conclude that

$$\mathcal{F}_{phys} = \text{span}\left(a_+^{+\,n_+}\, a_1^{+\,n_1}\, a_2^{+\,n_2}|0\rangle\right) \quad (5.41)$$

(ii) $\mathcal{F}_{phys}$ contains only states with semi-positive norm: $\langle\Psi|\Psi\rangle \geq 0$

Indeed

$$\| a_+^+|\Psi\rangle\|^2 = \langle\Psi| a_+ a_+^+ |\Psi\rangle$$
$$= \langle\Psi| a_+^+ a_+ |\Psi\rangle = 0 \quad (5.42)$$

and

$$\| a_1^{+\,n_1}\, a_2^{+\,n_2}|0\rangle\| > 0$$

with $[a_i^+, a_i] = +(2\pi)^3 2k^0\, \delta$ (5.43)

(iii) We identify two states $|\psi_1\rangle, |\psi_2\rangle$ with $\| |\psi_1\rangle - |\psi_2\rangle \| = 0$: every matrix element of an operator $\mathcal{O}(a_i^{(+)}, a_+^{(+)})$ vanishes $\langle \psi | \mathcal{O}(|\psi_1\rangle - |\psi_2\rangle)$ vanishes.

$\Rightarrow$ We define the physical Hilbert space as the space of equivalence classes

$$\mathcal{H} = \mathcal{F}_{phys} / \sim \qquad (5.44)$$

with $|\psi_1\rangle \sim |\psi_2\rangle$ for $\| |\psi_1\rangle - |\psi_2\rangle \| = 0$

We have for $|\psi\rangle \in \mathcal{H}$

$$\langle \psi | \psi \rangle > 0 \quad \text{for } |\psi\rangle \neq 0$$
$$(5.45)$$
$$\alpha_+ |\psi\rangle = 0$$

and hence $\langle \psi' | \partial_\mu F^{\mu\nu} | \psi \rangle = 0$

with $\langle \psi' | \partial_\mu F^{\mu\nu} | \psi \rangle = \langle \psi' | \partial^\nu \partial_\mu A^\mu | \psi \rangle = 0 \qquad (5.46)$

Feynman rules:

(i) Propagator: $x^0 > y^0$

$$\langle 0 | \; A_\mu(x) \, A_\nu(y) | 0 \rangle$$

$$= \langle 0 | \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2k^0} \, e^{-ikx + ik'y} \left[a_\mu(\vec{k}), a^\dagger_\nu(\vec{k}') \right] | 0 \rangle$$

$$= -\eta_{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \, e^{-ik(x-y)} \qquad (5.47)$$

$$\Rightarrow \quad \langle 0 | T \, A_\mu(x) \, A_\nu(y) | 0 \rangle = -\eta_{\mu\nu} \underbrace{D_F(x-y)}_{\text{scalar prop}}$$

(4.48)

or 

$$= -\frac{i \, \eta_{\mu\nu}}{k^2 + i\varepsilon}$$

(ii) final states / initial states

$$|\vec{k}, \varepsilon\rangle = \alpha^{+}(\vec{k})|0\rangle \qquad (4.49)$$

Note that $\alpha^{+} = \varepsilon^{*}_{\nu}\, a^{+\nu}$, eq. (5.34).

Hence we have

$$A_{\nu}(x)\Big|_{\text{annih}} |\vec{k}, \varepsilon\rangle = \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2k'^0} e^{-ik'x} a_{\nu}(\vec{k}')\alpha^{+}(\vec{k})|0\rangle$$

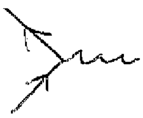
$$= \varepsilon^{*}_{\nu}(k) \qquad (4.50)$$

that is $\boxed{A|\vec{k}, \varepsilon\rangle := \varepsilon^{*}}$

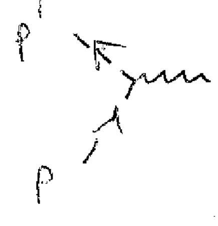
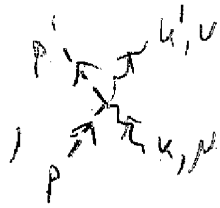
(4.51)

and $\langle \vec{k}, \varepsilon| A = \varepsilon$

(iii) Vertices:

(a) $\mathcal{L}_I = e\bar{\psi} \not{A} \psi$:  $= ie\gamma^{\mu}$ (4.52)

(b) $\mathcal{L}_I = D_{\nu}\phi (D^{\nu}\phi)^{*} - \partial_{\nu}\phi\, \partial^{\nu}\phi^{*}$ (4.53)

 $= -ie(p_{\nu} + p'_{\nu})$,  $= 2ie^2 \eta^{\mu\nu}$

Gauge independence & Feynman rules

We can add a longitudinal peace to the field A_μ without changing physics!

$$A_\mu \to A_\mu + \alpha\, \partial_\mu \frac{1}{\partial_\rho \partial^\rho} \partial_\nu A_\nu \qquad (5.54)$$

or in Lagrangian

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \qquad (5.55)$$

$\Rightarrow$ Propagator:

$$\langle 0 | T A_\mu A_\nu | 0 \rangle (p^2)$$

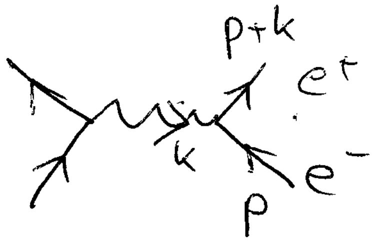
$=$ 

$$= -i \left(\frac{\eta_{\mu\nu}}{k^2 + i\varepsilon} - (1-\xi) \frac{k_\mu k_\nu}{(k^2 + i\varepsilon)^2} \right)$$

$$= \frac{-i}{k^2 + i\varepsilon} \left(\eta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2 + i\varepsilon} \right) \qquad (5.56)$$

ξ drops out of the scattering amplitudes

E.g.:



$$\simeq \langle 0|T A_\mu A_\nu|0\rangle(k)\, \bar{v}(p+k)\,\gamma^\nu u(p) \qquad (5.57)$$

We use:

$$\xi k_\nu k_\mu\, \bar{v}(p+k)\,\gamma^\mu u(p)$$

$$= \xi k_\nu\, \bar{v}(p+k)\, \not{k}\, u(p)$$

$$= \xi k_\nu \bar{v}(p+k)\left(\not{k} + \not{p} - \not{p}\right) u(p)$$

p. 116: $\underbrace{(\not{p}-m)u(p)}_{=0}$ $\Rightarrow$ $= \xi k_\nu\, \bar{v}(p+k)\left(\not{k} + \not{p} - m\right) u(p)$

$\downarrow$

$\underbrace{\bar{v}(p+k)(\not{p}+\not{k}-m)}_{=0}$ $\quad = 0$ $\qquad \square$

(5.58)

Gauge invariant observables:

e.g. E and B - fields

$$E^i = -F^{0i} = -\left(\partial^0 A^i - \partial^i A^0\right)$$

and $\qquad (5.59)$

$$B^i = \varepsilon^{ijk} F_{jk}$$

They read

$$\vec{E} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0}\, ik^0 \Big\{ \left(\vec{a} - \frac{\vec{k}}{k^0} a^0\right) e^{-ikx} - \left(\vec{a}^{\,+} - \frac{\vec{k}}{k^0} a^{0+}\right) e^{ikx} \Big\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0}\, ik^0 \Big\{ \left(\varepsilon_1 \alpha_1 + \varepsilon_2 \alpha_2\right) e^{-ikx} - \left(\varepsilon_1 \alpha_1^+ + \varepsilon_2 \alpha_2^+\right) e^{ikx} \Big\}$$

phys. pola.

$$\underbrace{- \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0}\, ik^0 \Big\{ \frac{\vec{k}}{k^0} \alpha_+ e^{-ikx} - \frac{\vec{k}}{k^0} \alpha_+^+ e^{ikx} \Big\}}_{\sim 0} \qquad (5.60)$$

Analogously

$$B^i(x) = \varepsilon^{ijk} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} ik^j \left\{ \varepsilon^k \cdot \alpha \, e^{-ikx} + \varepsilon^k \alpha^\dagger e^{ikx} \right\} \quad (5.61)$$

It follows that only $\alpha_{1/2}$ and α_+ appear in $\vec{E}$ and $\vec{B}$. Sandwiched between physical states $|\psi\rangle \in \mathcal{H}$, α_+ drops out.

The Hamiltonian reads, with $\pi^i = E^i$,

$$H = \int d^3x \left\{ \vec{\pi} \cdot \partial_0 \vec{A} + \underbrace{\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{1/2 (\vec{E}^2 - \vec{B}^2)} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + \underbrace{\vec{\nabla}}_{\vec{\nabla}\vec{E}=0} (\vec{E} A_0) \right\}$$

$$= \frac{1}{2} \int d^3x \left\{ \vec{E}^2 + \vec{B}^2 \right\} \quad (5.62)$$

where we have used

$$\vec{\nabla}\vec{E} = (-\partial^0 \partial^i A^i + \partial^{i2} A^0)$$

$$\partial_\mu A^\mu = 0 \longrightarrow \quad = (-\partial^{02} + \partial^{i2}) A^0 = 0$$

We insert the E-B-field operators eqs. (5.60), (5.61) arrive at

$$p^0 = H \simeq \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} k^0 \sum_{i=1}^{2} \left(\alpha_i \alpha_i^+ + \alpha_i^+ \alpha_i\right)$$

$$\simeq \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} k^0 \sum_{i=1}^{2} \alpha_i^+(\vec{k}) \alpha_i(\vec{k}) \qquad (5.63)$$

where we have dropped the α_+-terms, and, in the second line, the vacuum terms.

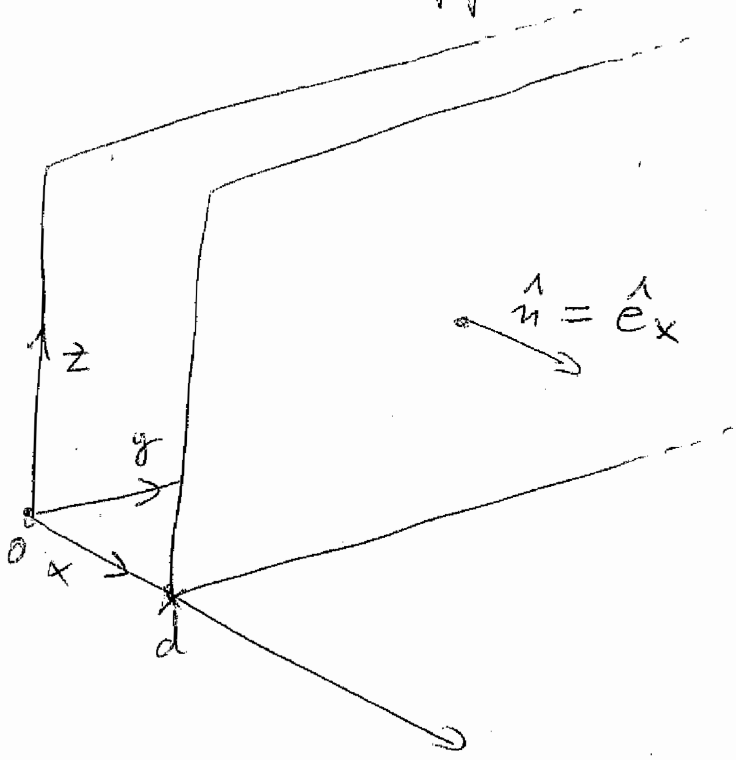
Similarily we get for $\vec{P}$:

$$\vec{P} = \int d^3x \; \vec{E} \times \vec{B}$$

$$\simeq \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2k^0} \vec{k} \sum_{i=1}^{2} \alpha_i^+(\vec{k}) \alpha_i(\vec{k}) \qquad (5.64)$$

where we have dropped the vacuum terms.

Casimir effect:



Exp.: Lamoreaux et al '97.
PRL 78 (1997) 5

Mohideen, Roy '98
PRL 81 (1998) 4549

Solution for $\vec{E}, \vec{B}$: plane waves

$$\vec{E} \simeq \underbrace{\vec{\varepsilon}}_{\text{polarisation}} e^{\pm ikx} \,, \quad \vec{B} \simeq \vec{k} \times \vec{E} \qquad (5.65)$$

with boundary conditions:

$$\hat{n} \times \vec{E}\Big|_{x=0,d} = 0$$

$$\hat{n} \times \vec{B}\Big|_{x=0,d} = 0 \qquad (5.66)$$

The electric (magnetic) fields parallel to the plates vanish on the plates!

Solution to eq (5.66):

$$\vec{E} \simeq \vec{\varepsilon} \sin k_x x \cdot e^{i(k_y y + k_z z - k^0 t)} \qquad (5.67)$$

with $k^0 = \sqrt{\vec{k}^2}$

$$k_x = n\pi/d \ , \quad n = 1, 2, \ldots \qquad (5.68)$$

We compute the ground state energy: see eq. (5.6

$$\langle 0|H|0\rangle_d = \frac{1}{2}\frac{1}{d}\sum_{n=1}^{\infty}\sum_{i=1}^{2}\int \frac{d^2 k_{\parallel}}{(2\pi)^2}\frac{1}{2}\langle 0|\alpha_i(\vec{k})\alpha_i^{\dagger}(\vec{k})|0\rangle \qquad (5.69)$$

vacuum state dep. on distance d

with $\int_{\mathbb{R}} \frac{dk_x}{(2\pi)} \to \frac{1}{d}\sum_{n=1}^{\infty}$ and $e^{ik_x x} \to \sin k_x x$

with k_x in (5.68)

and $\vec{k}_{\parallel} = (k_y, k_z)$.

The com. relat. eq. (5.26) of the α_i's now reads

$$[\alpha_i(\vec{k}), \alpha_j^{\dagger}(\vec{k}')] = (2\pi)^2\, 2k^0\, \delta^2(\vec{k}_{\parallel} - \vec{k}'_{\parallel}) \cdot \underbrace{2\pi\, \delta(k_x - k'_x)}_{\text{strictly speaking } d\,\delta\ldots}\, \delta_{ij} \qquad (5.70)$$

We conclude

$$\langle 0|H|0\rangle_d = \frac{1}{d}\sum_{n=1}^{\infty}\int\frac{d^2k_{\parallel}}{(2\pi)^2}\sqrt{\vec{k}_{\parallel}^{\,2}+\left(\frac{n\pi}{d}\right)^2}\,\underbrace{\left[(2\pi)^2\delta^2(0)\right]\cdot d}_{A\cdot d}$$

$$\text{with } (2\pi)^2\delta^2(0) = \int_A d^2x\, e^{-i(k_{\parallel}-k_{\parallel})x} \tag{5.71}$$

where A is the (infinite) area of the plate.

$$\Rightarrow \langle 0|H|0\rangle_d = A\sum_{n=1}^{\infty}\int\frac{d^2k_{\parallel}}{(2\pi)^2}\sqrt{\vec{k}_{\parallel}^{\,2}+\left(\frac{n\pi}{d}\right)^2} \tag{5.72}$$

We do not have to worry about the area factor, which is well-defined for finite plates (which then makes the formula an approximate one).

The other infinity comes from large momenta (UV). There, however, it is the by now common infinite vacuum energy which we may subtract.

In order to have well-defined quantities, we cut-off or regularise large momenta or high energies in eq. (5.72).

$$\langle 0|H|0\rangle_d \longrightarrow \hbar \sum_n \int \frac{d^2k_\parallel}{(2\pi)^2} \sqrt{\vec{k}_\parallel^2+\left(\frac{n\pi}{d}\right)^2} \cdot r_\Lambda\left(\vec{k}_\parallel^2+\left(\frac{n\pi}{d}\right)^2\right) \tag{5.73}$$

with $r_\Lambda(x \gg \Lambda^2) \to 0$, $r_\Lambda(x \ll \Lambda^2) \to 1$.

and hence

$$\boxed{\begin{aligned} E_{d,r_\Lambda} &= \langle 0|H|0\rangle_{d,r_\Lambda} \\ &= \hbar/_{2\pi} \sum_{n=1}^{\infty} R_\Lambda(n) \end{aligned}} \tag{5.74}$$

with $R_\Lambda(n) = \int_0^\infty dk_\parallel k_\parallel \sqrt{k_\parallel^2+\left(\frac{n\pi}{d}\right)^2}\; r_\Lambda\left(k_\parallel^2+\left(\frac{n\pi}{d}\right)^2\right)$

This energy cannot be measured; what can be measured, are energy differences. To that end we take the infinite distance limit, $d\to\infty$.

Then, $E_{d\to\infty, r_1}$ tends towards the standard (regularised) vacuum energy, which we have set to zero. The energy difference of the situation without plates, $d=\infty$, and that with plates, is

$$\boxed{\begin{aligned}\Delta E_{d,r_1} &= E_{d,r_1} - E_{\infty,r_1}\cdot\frac{V_d}{V_\infty}\\ &= \frac{\hbar}{2\pi}\left[\sum_{n=1}^{\infty} R_1(n) - \int_0^\infty dn\, R_1(n) + \frac{1}{2}R_1(0)\right]\end{aligned}} \qquad (5.75)$$

The integral in (5.75) can be turned into a sum (Euler-Maclaurin formula):

$$\int_0^\infty dn\, R_1(n) = \sum_{n=1}^{\infty}\left\{R_1(n) + \frac{1}{(2n)!}B_{2n}\, R_1^{(2n-1)}(0)\right\} + \frac{1}{2}R_1(0) \qquad (5.76)$$

with Bernoulli numbers B_{2n}.

We finally get

$$\boxed{\Delta E_{d,r_1} = \frac{\hbar}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n} R_1^{(2n-1)}(0)} \qquad (5.77)$$

and with $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$

$$\Delta E_{d,r_1} = \frac{\hbar}{2\pi} \left\{ -\frac{1}{12} R_1^{(1)}(0) + \frac{1}{720} R_1^{(3)}(0) + \ldots \right\} \qquad (5.78)$$

We use $k = \sqrt{k_\parallel^2 + \left(\frac{\pi n}{d}\right)^2} \Rightarrow dk\, k = dk_\parallel\, k_\parallel$

$$R_1(n) = \int_{n\pi/d}^{\infty} dk\, k^2\, r_1(k^2) \qquad (5.79)$$

Thence:

$$R_1^{(1)}(n) = -\pi/d \left(\frac{n\pi}{d}\right)^2 r_1\left(\left(n\pi/d\right)^2\right)$$

$$R_1^{(2)}(0) = 0$$

$$R_1^{(3)}(0) = -2\left(\pi/d\right)^3 \qquad (5.80)$$

and $R_1^{(i>3)}(0)$ depend on r_1 and $R_1^{(i>3)} \sim \left(\frac{1}{\Lambda d}\right)^{i-3}$

It follows for $\Lambda \to \infty$: $\Delta E_d = \Delta E_{d,1}$

$$\boxed{\Delta E_d = -\frac{\pi^2}{720} \hbar / d^3} \qquad (5.81)$$

Remarks:

(i) The energy difference ΔE_d is finite and independent of the regularisation procedure!

(ii) The Force/area is computed from the energy-density $\Delta \mathcal{E}_d = \Delta E_d / \cancel{L} = -\frac{\pi^2}{720} \, 1/d^3$

$$\Rightarrow \text{Force/area } f = -\frac{\partial \Delta \mathcal{E}_d}{\partial d}$$

$$= -\frac{\pi^2}{240} \frac{1}{d^4} (\hbar c)$$

$$\Rightarrow \boxed{f \simeq -1.3 \cdot 10^{-27} \, \mathrm{Pa} \, \frac{m^4}{d^4}}$$

(5.82)