7.2 QED

Action of QED: (p. 163, only with electron)

$$S_{QED}[A,\psi] = \int d^4x\, \overline{\psi}_0 (i\not{D} - m_0)\psi_0$$
$$-\frac{1}{4}\int d^4x\, F_{\mu\nu}(A_0)F^{\mu\nu}(A_0) - \frac{1}{\xi_0}\int d^4x \left(\partial_\mu A_0^\mu\right)^2 \tag{7.40}$$

with $D_\mu = \partial_\mu - ie_0 A_{0\mu}$, and $\psi_0 = \psi_{0e}$.

The action is gauge invariant under

$$A_{0\mu} \to A_{0\mu} + \frac{1}{e_0}\partial_\mu \alpha$$
$$\psi_0 \to e^{i\alpha}\psi_0 \tag{7.41}$$

of the bare fields $A_{0\mu}$ and ψ_0 (see p. 164).

We introduce renormalised fields & parameters:

$$A_{0\mu} = Z_A^{1/2} A_\mu$$
$$\psi_0 = Z_\psi^{1/2}\psi \tag{7.42}$$
$$e_0 = Z_e\, e$$
$$m_0 = Z_m\, m. \qquad [\xi_0 = Z_\xi\, \xi]$$

It can be shown, that gauge symmetry enforces the relation

$$\mu \frac{d}{d\mu}\left(Z_A^{1/2} Z_e\right) = 0, \qquad (7.43)$$

that is, $\mu \frac{d}{d\mu}(e A_\mu) = 0$. This and similar relations for correlation fcts. are called Ward-Takahashi identities (WTIs) and will be subject of QFT II.

Here we proceed with a heuristic argument for eq. (7.43):

(a) <u>Physical</u> gauge invariance should apply to renormalized quantities, so the covariant derivative should read

$$D_\mu = \partial_\mu - i e A_\mu \qquad (7.44)$$

which implies eq. (7.43).

(b) We have gauge-fixed the <u>bare</u>, <u>classical</u> action eq. (7.40). The argument in (a) only holds if this simple additive structure holds also on quantum level. To that end we evaluate

$$\langle S_{QED}[A^\alpha, \psi^\alpha] - S_{QED}[A,\psi]\rangle\Big|_{O(\alpha)}$$

$$= -\frac{1}{\xi} \int d^4x \underbrace{\langle \partial_\mu A^\mu\rangle}_{0} \partial_\rho \partial^\rho \alpha = 0 \tag{7.45}$$

! no quantum flucs. to gauge fixing !

! Heuristics !

We conclude that for general linear gauge fixings eq. (7.43) holds, and $Z_\xi = Z_A$.

In turn, for non-linear gauge fixings and for non-Abelian gauge theories (strong & weak forces) eq. (7.43) fails.

[Slavnov-Taylor identities / BRST in QFT II]

Feynman rules in terms of renormalized quantities: see p. 166 (and $Z_\xi = Z_A$ with WTI)

Props.:

$$\Big[\xrightarrow[p]{\quad}\Big]^{-1} = \frac{1}{i} Z_\psi (\not p - Z_m m)$$

$$= \left[i \frac{\not p + m}{p^2 - m^2} \right]^{-1} - \;—\!\otimes\!— \tag{7.46}$$

$$—\!\otimes\!— \;=\; -i(1 - Z_\psi)\not p + i(1 - Z_\psi Z_m) m$$

$$\Big[\underset{k}{\sim\sim}\Big]^{-1}_{\mu\nu} = i Z_A \left(k^2 \eta_{\mu\nu} - k_\mu k_\nu \left(1 - \frac{1}{Z_A \xi}\right)\right)$$

$$= \left[-\frac{i}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\right]^{-1} - \sim\!\otimes\!\sim \tag{7.47}$$

$$\sim\!\otimes\!\sim \;=\; i(1 - Z_A)(k^2 \eta_{\mu\nu} - k_\mu k_\nu)$$

↑ only transversal modes get renormalized

Vertex:

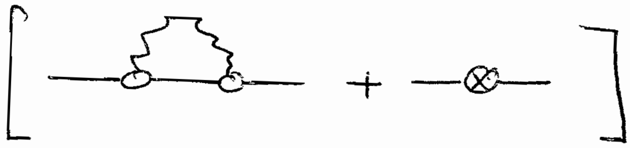
$$\text{vertex}_\mu = i Z_\psi Z_A^{1/2} Z_e e$$

$$= i e \gamma_\mu + \text{vertex}_\otimes \tag{7.48}$$

$$\text{vertex}_{\otimes\,\mu} = -i e \gamma_\mu (1 - Z_\psi Z_A^{1/2} Z_e)$$

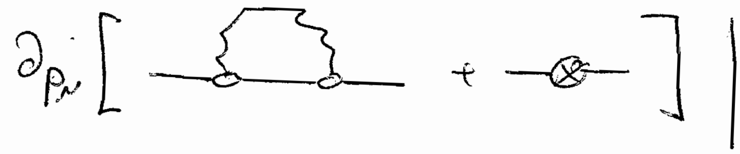
Renormalisation at one loop:

As in the scalar theory we can compute the mass correction via computing



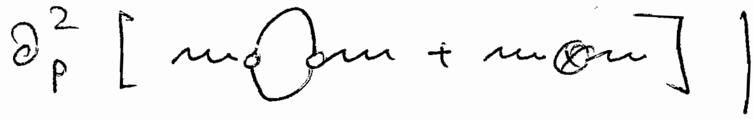
at $p^2 = m^2$

, the wave function renormalisations Z_ψ, Z_A via



$\Big|_{p^2=\mu^2}$

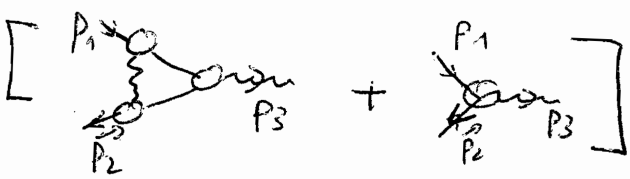
and



$\Big|_{p^2=\mu^2}$

simple example see p. 198a

and the coupling correction via the above computations, giving us $Z_{A/\psi}$ and Z_m, and



$p_i^2 = \mu^2$

How to project on the Z's?

Simple example: $Z_4(\not p - Z_m m)$

(1) $\partial_{p_\nu} Z_4(\not p - Z_m m)$

$$= Z_4 \gamma^\nu$$

(2) $\frac{1}{4d} \mathrm{Tr}\, \gamma_\nu \frac{\partial}{\partial p_\nu}\left(Z_4(\not p - Z_m m)\right) = Z_4 \mathrm{Tr}\, \gamma_\nu \gamma^\nu \frac{1}{4d}$

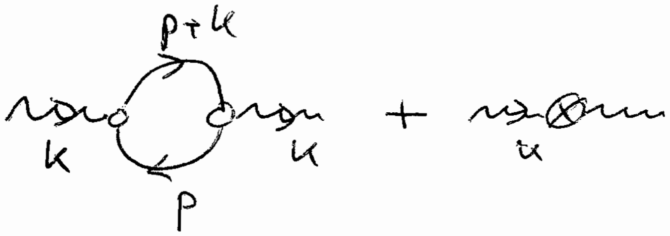
$$= Z_4$$

(3) $\frac{1}{4} \mathrm{Tr}\; Z_4(\not p - Z_m m)\Big|_{p=0} = \left(\frac{\mathrm{Tr}\, \mathbb{1}}{4}\right) Z_4 Z_m m$

$$= Z_4 Z_m m$$

Here we compute the renormalised coupling by using the relations between $Z_A^{1/2}$ and Z_e:

Vacuum polarisation in dim. reg.:

$$i\Pi_{\mu\nu}(k) = \Bigg[$$  $$\Bigg] \qquad (7.49)$$

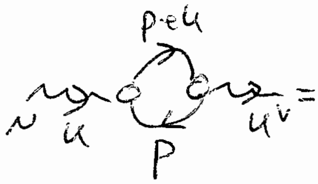
Feynman rules p. 198/166

in particular: no mass term

$$i\Pi_{\mu\nu}(k) = i\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k) \qquad \Pi(k) = \frac{1}{d-1}\Pi_\mu{}^\mu$$

gauge invariance $\qquad \eta_\mu{}^\mu = d$

(7.50)

and  $$= -e^2(\bar\mu^2)^{\frac{4-d}{2}} \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\, \frac{\not p + m}{p^2-m^2}\gamma_\mu \frac{\not p + \not k + m}{(p+k)^2-m^2}\gamma_\nu$$

(7.51)

First we evaluate the Dirac trace in $\Pi_\mu{}^\mu$

$$\mathrm{tr}\,(\not p + m)\gamma_\mu(\not p + \not k + m)\gamma^\mu \qquad (7.52)$$

in $d = 4 - 2\varepsilon$ dimensions

In d dimensions we have $\mathrm{Tr}\,\mathbb{1} = 4$ (non-trivial, but consider e.g. $d=3$)

and with $\eta_\nu{}^\nu = d$,

$$\gamma_\nu \gamma^\nu = d \cdot \mathbb{1} \quad \text{and} \quad \underbrace{\gamma^\mu \not{p}}_{\{\gamma^\mu, \not{p}\} - \not{p}\gamma^\mu} \gamma_\nu = 2\not{p} - \gamma^\mu \gamma_\nu \not{p}$$

$$= (2-d)\not{p} \qquad (7.53)$$

With eqs. (7.53) the trace in eq. (7.52) is computed as

$$\mathrm{tr}\,(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)\gamma^\nu$$

$$= \mathrm{tr}\,((2-d)\not{p}+dm)(\not{p}+\not{k}+m) = 4\left[(2-d)\,p\cdot(p+k)+dm^2\right] \qquad (7.54)$$

Inserting eq. (7.54) in $\Pi(p)$ we arrive at

$$i\Pi(k) = \frac{1}{d-1}\, 4e^2 (\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d p}{(2\pi)^4} \frac{(d-2)\,p(p+k) - dm^2}{(p^2-m^2)((p+k)^2-m^2)} \qquad (7.55)$$

Further simplification: Feynman parameter

$$\frac{1}{A\cdot B} = \int_0^1 d\alpha \, \frac{1}{[\alpha A + (1-\alpha)B]^2} \qquad (7.56)$$

see also exercise sheet 11 for generalisations

It follows

$$i\Pi(k) = \frac{4e^2}{d-1}\int_0^1 d\alpha\,(\bar{\mu}^2)^{\frac{4-d}{2}}\int\frac{d^dp}{(2\pi)^d}\,\frac{(d-2)\,p(p+k)-dm^2}{\left[(1-\alpha)(p^2-m^2)+\alpha\left((p+k)^2-m^2\right)\right]}$$

(7.57)

We disentangle the loop momentum p and the external momentum k:

$$p \to p - \alpha k$$

$$\Rightarrow \frac{(d-2)\,p(p+k)-dm^2}{\left[p^2+2\alpha pk+\alpha k^2-m^2\right]^2} \longrightarrow \frac{(d-2)\left(p^2+\overset{\int d\Omega_d \,\to\, 0}{(1-2\alpha)k\cdot p}-\alpha(1-\alpha)k^2\right)-dm^2}{\left[p^2+\underbrace{\alpha(1-\alpha)k^2-m^2}_{-\Delta}\right]^2}$$

(7.58)

Hence

$$i\Pi(k) = \frac{4e^2}{d-1}\int_0^1 d\alpha\,(\bar{\mu}^2)^{\frac{4-d}{2}}\int\frac{d^dp}{(2\pi)^d}\,\frac{(d-2)\left(p^2-\alpha(1-\alpha)k^2\right)-dm^2}{\left[p^2-\Delta\right]^2}$$

(7.59)

Wick rotation (see p. 186, 186a)

$$\boxed{i\Pi(k) = -i\,\frac{4e^2}{d-1}\int_0^1 d\alpha\,(\bar{\mu}^2)^{\frac{4-d}{2}}\int\frac{d^dp}{(2\pi)^d}\,\frac{(d-2)\left(p^2-\alpha(1-\alpha)k^2\right)+dm^2}{\left[p^2+\Delta\right]^2}}$$

(7.60)

with $\Delta = \alpha(1-\alpha)\,k^2+m^2$

The integrals can be performed with the help of the integrals on p. 187a ($m^2 \to \Delta$)

rearrangement p. 203a

We get

$$\Pi(k) = -\frac{4e^2}{d-1}\frac{1}{(4\pi)^{d/2}}\int_0^1 d\alpha \left(\frac{\Delta}{\bar{\mu}^2}\right)^{-\varepsilon}\Big\{(d-2)\,\Gamma(-1+\varepsilon)\,\Delta$$

$$-(d-2)\,\Gamma[\varepsilon]\,(\Delta + \alpha(1-\alpha)\,k^2)$$

$$+\ d\ \Gamma[\varepsilon]\ m^2\Big\} \qquad (7.61)$$

Expansion in ε leads to (see p. 203b)

$$\boxed{\Pi(k) = -\frac{1}{3\pi}\frac{e^2}{4\pi}\,k^2\Big[-\frac{1}{\varepsilon} + \gamma - \ln 4\pi + 6\int_0^1 d\alpha\ \alpha(1-\alpha)\ln \Delta/\bar{\mu}^2\Big]} \qquad (7.62)$$

Remark: no term $\sim m^2$ reflects transversality

The integrand in eq. (7.60) is brought into the form used on p. 187a with

$$\int \frac{d^dp}{(2\pi)^d} \frac{(d-2)\left(p^2 - \alpha(1-\alpha)k^2\right) + d m^2}{\left[p^2 + \Delta\right]^2}$$

$$\int \frac{d^dp}{(2\pi)^d} (d-2) \frac{1}{p^2+\Delta} + \frac{(d-2)\left(-\Delta - \alpha(1-\alpha)k^2\right) + d m^2}{\left[p^2+\Delta\right]^2}$$

p. 187a, $n=1$ $n=2$

$$= (d-2)\,\Gamma(-1+\varepsilon)\,\Delta^{1-\varepsilon} + \left[(d-2)\left(-\Delta - \alpha(1-\alpha)k^2\right) + d m^2\right]\Gamma[\varepsilon]\,\Delta^{-\varepsilon}$$

Inserting this in eq. (7.60) leads to eq. (7.61).

(1) Terms in eq. (7.61) proportional to m^2

$$-m^2\frac{4e^2}{d-1}\frac{1}{(4\pi)^{d/2}}\int_0^1 d\alpha\left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon}\left[(d-2)\Gamma(-1+\varepsilon)+2\Gamma(\varepsilon)\right]$$

$$=-m^2\frac{4e^2}{d-1}\frac{1}{(4\pi)^{d/2}}\int_0^1 d\alpha\left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon}\Big[\underbrace{(d-2)\left(-\frac{1}{\varepsilon}+\gamma-1\right)+2/_\varepsilon-2\gamma}_{0+O(\varepsilon)}\Big]+O(\varepsilon)$$

(7.63)

with $\Gamma[\varepsilon]=\frac{1}{\varepsilon}-\gamma+O(\varepsilon)$ and $\Gamma[-1+\varepsilon]=-\frac{1}{\varepsilon}+\gamma-1+O(\varepsilon)$

(2) Terms in eq. (7.61) proportional to k^2

$$-k^2\frac{4e^2}{d-1}\frac{1}{(4\pi)^{d/2}}\int_0^1 d\alpha\left[(d-2)\Gamma(-1+\varepsilon)-2(d-2)\Gamma(\varepsilon)\right]\alpha(1-\alpha)$$

$$=-k^2\frac{4e^2}{3}\frac{1}{(4\pi)^2}\int_0^1 d\alpha\,\alpha(1-\alpha)(2-2\varepsilon)\left(1-\varepsilon\ln\Delta/_{\tilde{\mu}^2}+\varepsilon\ln4\pi\right)\left(1+\frac{2\varepsilon}{3}\right)$$

$$\cdot\left[-3/_\varepsilon+3\gamma-1\right]+O(\varepsilon)$$

$$=-k^2\frac{1}{3\pi}\frac{e^2}{4\pi}\left[-\frac{1}{\varepsilon}+\gamma-\ln4\pi+6\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\frac{\Delta}{\tilde{\mu}^2}\right]$$

(7.64)

β - function: (for $k^2/m^2 \gg 1$)

We take the 'momentum derivative' of $\ln(\bar{\pi}(k)/k^2)$ up to order e^2:

$$\beta(k) = -\frac{1}{2} k \frac{d}{dk} \ln\left(\bar{\pi}(k)/k^2\right)$$

$$= \frac{2}{\pi} \frac{e^2}{4\pi} \int_0^1 dx\,(1-x)$$

$$\Rightarrow \quad \boxed{\beta = \frac{1}{12\pi^2} e^2 + \mathcal{O}(e^4)}$$

Compare with ϕ^4-theory, p. 193, 194:

The β-fcts. have the same sign!

$\Rightarrow$ QED is UV-sick

How this can possibly be cured $\Rightarrow$ QFT II