

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T} \frac{d^3 p_3}{F (2\pi)^3} \frac{1}{2p_3^0} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2p_4^0} \times \sum_{\text{spins}} |\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle|^2$$

with incident particle flux F , and unit volume $V = 1$.

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with incident particle flux F , and unit volume $V = 1$.

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- S-Matrix:

$$S = T e^{ie \int d^4x A_\nu(x) : \bar{\psi} \gamma^\nu \psi(x) :}$$

with

$$\bar{\psi} \gamma^\nu \psi(x) = \bar{\psi}_e \gamma^\nu \psi_e(x) + \bar{\psi}_\mu \gamma^\nu \psi_\mu(x)$$

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- S-Matrix:

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- Expansion of S-matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$:

$$\begin{aligned} & \langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle \\ &= \frac{(ie)^2}{2} \langle \mu^+(p_4) \mu^-(p_3) | T \int d^4x d^4x' A_\nu(x) A_\mu(x') \\ & \quad \times : \bar{\psi} \gamma^\nu \psi(x) :: \bar{\psi} \gamma^\mu \psi(x') : | e^+(p_2) e^-(p_1) \rangle + O(e^4) \end{aligned}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

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- states

$$| e^+(p_2) e^-(p_1) \rangle = b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle$$

- fermionic field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_{s=\pm 1/2} \{ e^{ipx} v_s(p) b^\dagger(\vec{p}) + e^{-ipx} u_s(p) a(\vec{p}) \}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- A_μ commutes with $\psi, \bar{\psi}$:

$$\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle$$

$$= \frac{(ie)^2}{2} \int d^4x d^4x' \langle 0 | T A_\nu(x) A_\mu(x') | 0 \rangle$$

$$\times \langle \mu^+(p_4) \mu^-(p_3) | T : \bar{\psi} \gamma^\nu \psi(x) :: \bar{\psi} \gamma^\mu \psi(x') : | e^+(p_2) e^-(p_1) \rangle + O(e^4)$$

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S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- A_μ commutes with $\psi, \bar{\psi}$:

$$\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle$$

$$= (ie)^2 \int d^4x d^4x' \langle 0 | T A_\nu(x) A_\mu(x') | 0 \rangle$$

$$\times \langle \mu^+(p_4) \mu^-(p_3) | : \bar{\psi}_\mu \gamma^\nu \psi_\mu(x) :: \bar{\psi}_e \gamma^\mu \psi_e(x') : | e^+(p_2) e^-(p_1) \rangle + O(e^4)$$

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- counting annihilation/creation operators $a^{(\dagger)}, b^{(\dagger)}$

$$\langle \mu^+(p_4) \mu^-(p_3) | : \bar{\psi}_\mu \gamma^\nu \psi_\mu(x) :: \bar{\psi}_e \gamma^\mu \psi_e(x') : | e^+(p_2) e^-(p_1) \rangle$$

$$= \langle \mu^+(p_4) \mu^-(p_3) | : \bar{\psi}_\mu \gamma^\nu \psi_\mu(x) : | 0 \rangle \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(x') : | e^+(p_2) e^-(p_1) \rangle$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : | e^+(p_2) e^-(p_1) \rangle$$

$$= \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle$$

- reminder

$$\psi(\mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} \sum_{s=\pm 1/2} \{ e^{ipx} v_s(p) b^\dagger(\vec{p}) + e^{-ipx} u_s(p) a(\vec{p}) \}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\begin{aligned} \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{\mathbf{p}}_2) a_e^\dagger(\vec{\mathbf{p}}_1) | 0 \rangle \\ = - \langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{\mathbf{p}}_2) | 0 \rangle \gamma^\mu \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{\mathbf{p}}_1) | 0 \rangle \end{aligned}$$

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$$\begin{aligned} \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle \\ = - \langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{p}_2) | 0 \rangle \gamma^\mu \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle \end{aligned}$$

- expectation value $\langle 0 | \psi | e^- \rangle$

$$\langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0 | a(\vec{p}) a_e^\dagger(\vec{p}_1) | 0 \rangle$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\begin{aligned} \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle \\ = - \langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{p}_2) | 0 \rangle \gamma^\mu \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle \end{aligned}$$

- expectation value $\langle 0 | \psi | e^- \rangle$

$$\langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0 | \{ a_e(\vec{p}), a_e^\dagger(\vec{p}_1) \} | 0 \rangle$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\begin{aligned} \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle \\ = - \langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{p}_2) | 0 \rangle \gamma^\mu \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle \end{aligned}$$

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- reminder

$$\{ a_s(\vec{p}), a_r^\dagger(\vec{p}_1) \} = (2\pi)^3 2p^0 \delta_{rs} \delta(\vec{p} - \vec{p}_1)$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\begin{aligned} \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle \\ = - \langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{p}_2) | 0 \rangle \gamma^\mu \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle \end{aligned}$$

- expectation value $\langle 0 | \psi | e^- \rangle$

$$\begin{aligned} \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0 | \{ a_e(\vec{p}), a_e^\dagger(\vec{p}_1) \} | 0 \rangle \\ &= e^{-ip_1 x'} u_e(p_1) \end{aligned}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\begin{aligned} \langle 0 | : \bar{\psi}_e \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle \\ = - \langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{p}_2) | 0 \rangle \gamma^\mu \langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle \end{aligned}$$

- expectation values $\langle 0 | \psi | e^- \rangle, \langle 0 | \bar{\psi} | e^+ \rangle$

$$\langle 0 | \psi_e(\mathbf{x}') a_e^\dagger(\vec{p}_1) | 0 \rangle = e^{-ip_1 x'} u_e(p_1)$$

$$\langle 0 | \bar{\psi}_e(\mathbf{x}') b_e^\dagger(\vec{p}_2) | 0 \rangle = e^{-ip_2 x'} \bar{v}_e(p_2)$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\langle 0 | : \bar{\psi}_e(\mathbf{x}') \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle$$

$$= -\bar{v}_e(p_2) \gamma^\mu u_e(p_1) e^{-i(p_1 + p_2) \cdot x'}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- further reduction

$$\langle 0 | : \bar{\psi}_e(\mathbf{x}') \gamma^\mu \psi_e(\mathbf{x}') : b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) | 0 \rangle$$

$$= -\bar{v}_e(p_2) \gamma^\mu u_e(p_1) e^{-i(p_1 + p_2) \cdot \mathbf{x}'}$$

- similarly

$$\langle 0 | b_\mu(\vec{p}_4) a_\mu(\vec{p}_3) : \bar{\psi}_\mu(\mathbf{x}) \gamma^\nu \psi_\mu(\mathbf{x}) : | 0 \rangle$$

$$= \bar{u}_\mu(p_3) \gamma^\nu v_\mu(p_4) e^{i(p_3 + p_4) \cdot \mathbf{x}}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle$$

$$\simeq -(ie)^2 \int d^4x d^4x' \langle 0 | T A_\nu(x) A_\mu(x') | 0 \rangle e^{i(p_3+p_4)x} e^{-i(p_1+p_2)x'} \\ \times \bar{u}_\mu(p_3) \gamma^\nu v_\mu(p_4) \bar{v}_e(p_2) \gamma^\mu u_e(p_1)$$

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$$\times \bar{u}_\mu(p_3) \gamma^\nu v_\mu(p_4) \bar{v}_e(p_2) \gamma^\mu u_e(p_1)$$

- photon propagator

$$\langle 0 | T A_\nu(x) A_\mu(x') | 0 \rangle = -ig_{\mu\nu} \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 + i\epsilon}$$

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- momentum conservation

$$\int d^4x d^4x' \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i(k-p_1-p_2)x} e^{i(k-p_3-p_4)x'}}{k^2 + i\epsilon} \\ = \frac{1}{s} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)$$

with $s = (p_1 + p_2)^2$, the square of the total energy

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle$$

$$\simeq \frac{ig_{\mu\nu}}{s} (ie)^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \bar{u}_\mu(p_3) \gamma^\nu v_\mu(p_4) \bar{v}_e(p_2) \gamma^\mu u_e(p_1)$$

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- reminder: differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{TF} \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2p_3^0} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2p_4^0}$$

$$\times \sum_{\text{spins}} |\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle|^2$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s, s', r, r'} |\bar{u}_{\mu, s}(p_3) \gamma^\nu v_{\mu, s'}(p_4) \bar{v}_{e, r}(p_2) \gamma_\nu u_{e, r'}(p_1)|^2$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\begin{aligned} & \frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2 \\ &= \frac{1}{2} \sum_{s,s'} \bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{\mu,s'}(p_4) \gamma^\rho u_{\mu,s}(p_3) \\ & \quad \times \frac{1}{2} \sum_{r,r'} \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1) \bar{u}_{e,r'}(p_1) \gamma_\rho v_{e,r}(p_2) \end{aligned}$$

with

$$[\bar{v}_r(p) \gamma_\nu u_{r'}(q)]^* = u_{r'}^\dagger(q) \gamma_\nu^\dagger \gamma^{0\dagger} v_r(p) = \bar{u}_{r'}(q) \gamma_\nu v_r(p)$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

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S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

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$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2} \sum_{s,s'} \bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{\mu,s'}(p_4) \gamma^\rho u_{\mu,s}(p_3)$$

- reminder: spin sums

$$\sum_{s=\pm 1/2} u_s(p) \bar{u}_s(p) = \not{p} + m, \quad \sum_{s=\pm 1/2} v_s(p) \bar{v}_s(p) = \not{p} - m$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

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- averaging/summation over spins in initial state/final state

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- single sum

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$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2} \sum_{s,s'} \bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{\mu,s'}(p_4) \gamma^\rho u_{\mu,s}(p_3) = \frac{1}{2} \text{Tr}(\not{p}_3 + m_\mu) \gamma^\nu (\not{p}_4 - m_\mu) \gamma^\rho$$

- reminder: spin sums

$$\sum_{s=\pm 1/2} u_s(p) \bar{u}_s(p) = \not{p} + m, \quad \sum_{s=\pm 1/2} v_s(p) \bar{v}_s(p) = \not{p} - m$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2} \sum_{s,s'} \bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{\mu,s'}(p_4) \gamma^\rho u_{\mu,s}(p_3) = \frac{1}{2} \text{Tr}(\not{p}_3 + m_\mu) \gamma^\nu (\not{p}_4 - m_\mu) \gamma^\rho$$

- similarly

$$\frac{1}{2} \sum_{s,s'} \bar{v}_{e,s}(p_2) \gamma_\nu u_{e,s'}(p_1) \bar{u}_{e,s'}(p_1) \gamma_\rho v_{e,s}(p_2) = \frac{1}{2} \text{Tr}(\not{p}_2 - m_e) \gamma_\nu (\not{p}_1 + m_e) \gamma_\rho$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\begin{aligned} & \frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2 \\ &= \frac{1}{4} \text{Tr} [(\not{p}_3 + m_\mu) \gamma^\nu (\not{p}_4 - m_\mu) \gamma^\rho] \text{Tr} [(\not{p}_2 - m_e) \gamma_\nu (\not{p}_1 + m_e) \gamma_\rho] \end{aligned}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\begin{aligned} & \frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2 \\ &= \frac{1}{4} \text{Tr} [(\not{p}_3 + m_\mu) \gamma^\nu (\not{p}_4 - m_\mu) \gamma^\rho] \text{Tr} [(\not{p}_2 - m_e) \gamma_\nu (\not{p}_1 + m_e) \gamma_\rho] \end{aligned}$$

- high energy limit

$$s \gg m_\mu^2, m_e^2$$

→ drop m_e, m_μ in traces

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- traces

$$\text{Tr} \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\rho = 4 (g^{\alpha\nu} g^{\beta\rho} + g^{\rho\alpha} g^{\nu\beta} - g^{\alpha\beta} g^{\nu\rho})$$

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- high energy limit revisited

$$p_1 p_3 = p_2 p_4 = \frac{S}{4}(1 - \cos \vartheta), \quad p_1 p_4 = p_2 p_3 = \frac{S}{4}(1 + \cos \vartheta)$$

with scattering angle

$$\cos \vartheta = \frac{\vec{p}_1 \vec{p}_3}{|\vec{p}_1| |\vec{p}_3|}$$

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3) \gamma^\nu v_{\mu,s'}(p_4) \bar{v}_{e,r}(p_2) \gamma_\nu u_{e,r'}(p_1)|^2$$
$$= \frac{s^2}{2} (1 + \cos^2 \vartheta)$$

- high energy limit revisited

$$p_1 p_3 = p_2 p_4 = \frac{s}{4} (1 - \cos \vartheta), \quad p_1 p_4 = p_2 p_3 = \frac{s}{4} (1 + \cos \vartheta)$$

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$$\cos \vartheta = \frac{|\vec{p}_1 \vec{p}_3|}{|\vec{p}_1| |\vec{p}_3|}$$

cross-section for $e^-e^+ \rightarrow \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T} \frac{d^3 p_3}{F (2\pi)^3} \frac{1}{2p_3^0} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2p_4^0} \times \sum_{\text{spins}} |\langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle|^2$$

with particle flux F , and unit volume $V = 1$.

- reminder: S-Matrix element

$$\begin{aligned} & \langle \mu^+(p_4) \mu^-(p_3) | S | e^+(p_2) e^-(p_1) \rangle \\ & \simeq \frac{i}{S} (ie)^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \bar{u}_\mu(p_3) \gamma^\nu v_\mu(p_4) \bar{v}_e(p_2) \gamma_\nu u_e(p_1) \end{aligned}$$

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

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- Fermi's trick

$$[(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)]^2 \\ = (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \int_{VT} d^4 x e^{ix((p_1 + p_2 - p_3 - p_4))} \\ = VT (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)$$

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

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- $\alpha = \frac{e^2}{4\pi}$

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3 p_3}{2p_3^0} \frac{d^3 p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4) (1 + \cos^2 \vartheta)$$

- high energy limit + CMS-system: $\vec{p}_1 + \vec{p}_2 = 0$, $p_1^0 + p_2^0 \simeq \sqrt{s}$

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

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- high energy limit + CMS-system: $\vec{p}_1 + \vec{p}_2 = 0, \quad p_1^0 + p_2^0 \simeq \sqrt{s}$
- differential cross-section $\frac{d\sigma}{d\Omega_3}$

$$\frac{d\sigma}{d\Omega_3} = \alpha^2 \frac{1}{2F} \int_0^\infty d|\vec{p}_3| |\vec{p}_3| \int \frac{d^3 p_4}{|\vec{p}_4|} \delta(\sqrt{s} - |\vec{p}_3| - |\vec{p}_4|) \delta(\vec{p}_3 + \vec{p}_4) (1 + \cos^2 \vartheta)$$

- Flux: $F = 2p_1^0 2p_2^0 \frac{|\vec{p}_1|}{p_1^0}$
($= |\vec{v}_A| 2E_A 2E_B$)

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3 p_3}{2p_3^0} \frac{d^3 p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4) (1 + \cos^2 \vartheta)$$

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$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{4} (1 + \cos^2 \vartheta)$$

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$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{4} (1 + \cos^2 \vartheta)$$

- total cross-section $\sigma = \int d\Omega_3 \frac{d\sigma}{d\Omega_3}$

$$\sigma_{\text{total}}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{4\pi\alpha^2}{3}$$