Quantum Field Theory 1
Exam

Problem 1: *Scalar field*
Consider the following action for two real scalar fields ϕ_1 and ϕ_2

$$S = \int d^4x \mathcal{L} = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \right\}.$$

a) Show that S is invariant under the infinitesimal transformation

$$\begin{aligned} \phi_1(x) &\rightarrow \phi_1(x) + \alpha\phi_2(x) \\ \phi_2(x) &\rightarrow \phi_2(x) - \alpha\phi_1(x). \end{aligned}$$

b) Using Noethers theorem or otherwise find the conserved current $j^\mu = (j^0, \vec{j})$ associated with this symmetry.

c) Suppose that the interaction term

$$\mathcal{L}_{\text{int}} = -\lambda\phi_1^4 - 2\sigma\phi_1^2\phi_2^2 - \lambda\phi_2^4$$

is added to the Lagrangian. For what values of λ and σ is j^μ still a conserved current?

Problem 2: *Dirac field*
The Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0,$$

where the gamma matrices in the chiral representation are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

with $\sigma^\mu = (\mathbb{1}_2, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i)$ and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the Pauli matrices.

a) Show that these matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$.

b) Show that the spinor $\psi(x)$ satisfies the Klein-Gordon equation.

c) Assume now that the field ψ is canonically quantized

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\left\{ a^s_{\vec{p}}\ u_s(\vec{p})\ e^{-ipx} + (b^s_{\vec{p}})^\dagger\ v_s(\vec{p})\ e^{ipx} \right\} \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\left\{ (a^s_{\vec{p}})^\dagger\ \bar{u}_s(\vec{p})\ e^{ipx} + b^s_{\vec{p}}\ \bar{v}_s(\vec{p})\ e^{-ipx} \right\}. \end{aligned}$$

What quantization relation do the operators $a^s_{\vec{p}}$, $(a^s_{\vec{p}})^\dagger$, $b^s_{\vec{p}}$ and $(b^s_{\vec{p}})^\dagger$ satisfy?

d) Consider the conserved charge

$$Q = \int d^3x \, \bar{\psi}\gamma^0\psi.$$

Show that

$$Q = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\left\{(a^s_{\vec{p}})^\dagger\, a^s_{\vec{p}} - (b^s_{\vec{p}})^\dagger\, b^s_{\vec{p}}\right\} + \text{const.}$$

You can use the relations

$$\bar{u}_s(\vec{p})\,\gamma^0\,u_r(\vec{p}) = \bar{v}_s(\vec{p})\,\gamma^0\,v_r(\vec{p}) = 2p^0\,\delta_{rs},$$
$$\bar{u}_s(\vec{p})\,\gamma^0\,v_r(-\vec{p}) = \bar{v}_s(\vec{p})\,\gamma^0\,u_r(-\vec{p}) = 0.$$

PROBLEM 3: *Feynman rules*

Consider a theory involving the real scalar fields Φ and φ with the Langrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - h\Phi\varphi\varphi.$$

a) Write down the Feynman rules in momentum space for this theory.

b) Consider a scattering experiment where two φ-particles with initial momenta $\vec{k}_1$ and $\vec{k}_2$ are scattered to final momenta $\vec{p}_1$ and $\vec{p}_2$. Show that to order h^2 there are 3 tree-level contributions to the scattering matrix. Draw the corresponding Feynman diagrams and determine the contribution to the scattering amplitude $i\mathcal{M}$.

PROBLEM 4: *Gauge symmetry*

The Lagrangian of a scalar field φ of mass m and charge e, interacting with the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi - m^2\varphi^*\varphi,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\varphi = \partial_\mu\varphi - ieA_\mu\varphi$.

a) Show that this Lagrangian has a gauge symmetry.