Problems for Quantum Field Theory 1

1. Sheet

Suggested reading before solving these problems: Chapters 2.1 and 2.2. in the script and/or Chapters 2.1 to 2.4 of *Peskin & Schroeder*.

Problem 1: *Lagrangian "String Theory"*

Consider a series of one-dimensional coupled oscillators y_i, $i = 1, ..., N$ with distance a, boundary conditions $y_0 = y_{N+1} = 0$, and the Lagrange function

$$L = \sum_{i=1}^{N} \frac{1}{2}\, m\, \dot{y}_i^2 \;-\; \sum_{i=0}^{N} \frac{1}{2}\, t\, \left(\frac{y_{i+1} - y_i}{a}\right)^2 .$$

Show that the Lagrange function becomes that of a (clamped) string

$$L = \int_0^R dx \left\{ \frac{1}{2}\,\sigma \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\,\tau \left(\frac{\partial y}{\partial x}\right)^2 \right\}$$

in the limit $N \to \infty, a \to 0$ with $R = N \cdot a$ fixed. Here $\sigma = m/a$ is the mass per unit length and $\tau = t/a$ is the string tension. By expanding the displacement as a Fourier expansion in the form

$$y(x,t) = \sqrt{\frac{2}{R}} \sum_{n=1}^{\infty} q_n(t)\;\sin\left(\frac{n\pi x}{R}\right)$$

show that

$$L = \sum_{n=1}^{\infty} \left\{ \frac{1}{2}\,\sigma\,\dot{q}_n^2 - \frac{1}{2}\,\tau \left(\frac{n\pi}{R}\right)^2 q_n^2 \right\}.$$

Use the variational principle with this form of the Lagrangian to obtain the Euler-Lagrange equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0.$$

Hence show that the string is equivalent to an infinite set of harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{\tau}{\sigma}}\frac{n\pi}{R}.$$

What happens in the limit $R \to \infty$?

Problem 2: *Complex scalar field*
Consider the following action for a complex scalar field

$$S = \int d^4x \mathcal{L} = \int d^4x \left\{ \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \right\}.$$

It is easiest to consider ϕ and ϕ^* as independent, rather than the real and imaginary parts of ϕ.

a) Derive the Euler-Lagrange equations for ϕ and ϕ^*.

b) Show that S is invariant under the infinitesimal transformation

$$\begin{aligned} \phi(x) &\rightarrow (1 + i\alpha)\, \phi(x) \\ \phi^*(x) &\rightarrow (1 - i\alpha)\, \phi^*(x). \end{aligned} \tag{1}$$

c) Derive an expression for the Noether current $j^\mu = (j^0, \vec{j})$ associated with this symmetry transformation and show that it is conserved for fields ϕ, ϕ^* that satisfy the Euler-Lagrange equations.

d) Show that the invariance of S under infinitesimal space and time translations leads to four conserved currents. Give interpretations for the components of the energy-momentum tensor

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \partial_\nu \phi^* - \mathcal{L}\, \delta^\mu{}_\nu$$

and derive explicit expressions.