Problems for Quantum Field Theory 1
10. Sheet

This exercise is based on chapter 5.2 in the script and chapter I.8 of the book *Quantum Field Theory in a Nutshell* of A. Zee. [1]

Problem 1: *Vacuum energy*
Consider a free scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The Hamiltonian density for this theory is given by

$$H = \frac{1}{2}\int d^3x \left\{\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right\}.$$

Show that its vacuum expectation value (without normal ordering) can be written after restoring factors of $\hbar$ and c as

$$\langle 0|H|0\rangle = V\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2}\hbar\omega_p, \tag{1}$$

with $\hbar\omega_p = \sqrt{c^2p^2 + c^4m^2}$. For the electromagnetic field two things change: One has now two polarizations which leads to an additional factor 2 and the dispersion relation is $\hbar\omega_p = \sqrt{c^2\vec{p}^2}$. In the lecture we have used normal ordering for setting the vaccum energy in (1).

Problem 2: *Casimir force*
Can we somehow measure the vacuum energy? Consider the following setup. Two "perfectly" conducting plates of infinite extend (and zero thickness) are placed parallel to each other in a distance d. Due to the boundary conditions for the electromagnetic field, the wave vector can only take the values $(\frac{\pi n}{d}, k_y, k_z)$ with n an integer. The energy E between the plates is for an area $\mathcal{A}$

$$E = \langle 0|H|0\rangle = \hbar c\mathcal{A}\sum_n \int \frac{dk_y dk_z}{(2\pi)^2}\sqrt{\left(\frac{\pi n}{d}\right)^2 + k_y^2 + k_z^2}.$$

We could now calculate the force between the plates by varying the distance of the second plate with respect to the first one. However, we must also take the change in the field outside the plates into account. It is therefore useful to introduce a third plate with distance L to the first and distance $L - d$ to the second one. We vary only the position of the second plate in the middle and therefore do not change the field outside our experiment.

[1] This is a nice book to read in during Christmas holidays, another one is *QED: Die seltsame Theorie des Lichts und der Materie* by R.P. Feynman.

For simplicity we switch now to a field theory in $1+1$ space-time dimensions such that we do not have to integrate over k_x and k_y. The energy of the electromagnetic field is then given by

$$E = f(d) + f(L-d)$$

with

$$f(d) = \hbar c \sum_{n=1}^{\infty} \frac{\pi n}{d} = \frac{\hbar c \pi}{d} \sum_{n=1}^{\infty} n.$$

What should we do with this quite divergent expression? Thinking about the physics we realize that large n correspond to large values of the wavevector and therefore large energies of the involved photons. In reality, our plates will become invisible for photons with wavelength much smaller than the distance a between atoms in the plate. It is therefore reasonable to cut of the sum for $n\pi/d \gg a^{-1}$ by introducing a factor $e^{-an\pi/d}$. Show that we now obtain a finite expression

$$f(d) = \frac{\hbar c \pi}{d} \sum_{n=1}^{\infty} n e^{-an\pi/d} = \frac{\hbar c \pi}{d} \frac{e^{a\pi/d}}{(e^{a\pi/d}-1)^2},$$

and that for small a this becomes

$$f(d) = \frac{d\hbar c}{\pi a^2} - \frac{\hbar c \pi}{12 d} + \mathcal{O}(a^2).$$

We can now calculate the force F with

$$F = -\frac{\partial E}{\partial d} = f'(L-d) - f'(d).$$

Show that for $a \to \infty$ and $L \gg d$ the force reads

$$F = -\frac{\hbar c \pi}{12 d^2}.$$