Problems for Quantum Field Theory 1

11. Sheet

Suggested reading before solving these problems: Chapter 6.1-6.2 in the script and/or chapter 7.5 of *Peskin & Schroeder*.

Problem 1: *Dimensional regularization*

Dimensional regularization is a method to regularize expressions that formally diverge in $d = 3+1$ dimensions. One takes d to be a real number, for example $d = 4-2\epsilon$. A typical example is a (Euclidean or Wick-rotated) integral of the form

$$\int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2+m^2)^n} = \frac{1}{2}\int \frac{d\Omega_d}{(2\pi)^d}\; 2\int dq \frac{q^{d-1}}{(q^2+m^2)^n}. \qquad (1)$$

a) The first integral in Eq. (1) contains the area of a unit sphere in d dimensions. By using $(\sqrt{\pi})^d = \left(\int dx\, e^{-x^2}\right)^d = \int d^dx\, e^{-\sum_{i=1}^d x_i^2}$, show that

$$\frac{1}{2}\int \frac{d\Omega_d}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)},$$

with the Gamma function $\Gamma(x) = \int_0^\infty dt\, t^{x-1}e^{-t}$.

b) By substituting $x = m^2/(q^2+m^2)$ show that the remaining integral in Eq. (1) can be written as

$$2\int dq \frac{q^{d-1}}{(q^2+m^2)^n} = \left(\frac{1}{m^2}\right)^{n-\frac{d}{2}} \int_0^1 dx\, x^{n-\frac{d}{2}-1}(1-x)^{\frac{d}{2}-1}.$$

Use now the formula $\int_0^1 dx\, x^{\alpha-1}(1-\alpha)^{\beta-1} = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ to show

$$\int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+m^2)^n} = \frac{1}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\left(\frac{1}{m^2}\right)^{n-\frac{d}{2}} \frac{\Gamma(n-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(n)}. \qquad (2)$$

c) Derive from Eq. (2)

$$\int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+m^2+2q\cdot p)^n} = \frac{\Gamma(n-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(n)}\frac{1}{(m^2-p^2)^{n-\frac{d}{2}}},$$

and use this together with properties of the Gamma function to show

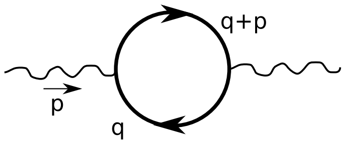
$$\int \frac{d^dq}{(2\pi)^d}\frac{q^\mu}{(q^2+m^2+2q\cdot p)^n} = -\frac{\Gamma(n-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(n)}\frac{p^\mu}{(m^2-p^2)^{n-\frac{d}{2}}}$$

and

$$\begin{aligned}\int \frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{(q^2+m^2+2q\cdot p)^n} &= \frac{1}{(4\pi)^{d/2}\Gamma(n)}\\ &\cdot\left[p^\mu p^\nu \frac{\Gamma(n-\frac{d}{2})}{(m^2-p^2)^{n-\frac{d}{2}}} + \frac{1}{2}\delta^{\mu\nu}\frac{\Gamma(n-\frac{d}{2}-1)}{(m^2-p^2)^{n-\frac{d}{2}-1}}\right].\end{aligned}$$

Problem 2: *Vacuum polarization of QED*

In this problem we calculate a one-loop contribution to the photon propagator of the form below.



a) Use the Feynman rules of QED to show that this amplitude reads

$$i\mathcal{M} = \epsilon^*_\mu(p)\,(i\Pi^{\mu\nu}(p))\,\epsilon_\nu(p)$$

with

$$i\Pi^{\mu\nu}(p) = -(-ie)^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\left[\gamma^\mu \frac{i(\not{q}+m)}{q^2-m^2}\gamma^\nu \frac{i(\not{q}+\not{p}+m)}{(q+p)^2-m^2}\right].$$

b) Use now the trace identity

$$\mathrm{tr}\ \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho = 4(\eta^{\mu\nu}\eta^{\sigma\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}),$$

the Feynman trick

$$\frac{1}{AB} = \int d\alpha \frac{1}{(\alpha A + (1-\alpha)B)^2},$$

an appropriate shift in the integration variable and Wick rotation to show

$$\begin{aligned} i\Pi^{\mu\nu}(p) &= -4ie^2 \int_0^1 d\alpha \int \frac{d^dq}{(2\pi)^d} \\ &\cdot \frac{-\frac{2}{d}\eta^{\mu\nu}q^2 + \eta^{\mu\nu}q^2 - 2\alpha(1-\alpha)p^\mu p^\nu + \eta^{\mu\nu}(m^2+\alpha(1-\alpha)p^2)}{(q^2+\Delta)^2} \end{aligned}$$

with $\Delta = m^2 - \alpha(1-\alpha)q^2$.

c) You can now use the results of problem 1 as well as the expansion ($d = 4-2\epsilon$)

$$\Gamma(2-\frac{d}{2}) = \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon),$$

with $\gamma \approx 0.5772$ the Euler-Mascheroni constant to show that $\Pi^{\mu\nu}(p)$ is of the form

$$i\Pi^{\mu\nu}(p) = (p^2\eta^{\mu\nu} - p^\mu p^\nu)i\Pi(p^2)$$

with

$$\begin{aligned} \Pi(p^2) &= -\frac{8e^2}{(4\pi)^{d/2}} \int_0^1 d\alpha\, \alpha(1-\alpha)\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \\ &\overset{d\to 4}{\to} -\frac{e^2}{2\pi^2}\int_0^1 d\alpha\, \alpha(1-\alpha)\left(\frac{1}{\epsilon} - \ln\Delta - \gamma\right). \end{aligned}$$