Problems for Quantum Field Theory 1
2. Sheet

Suggested reading before solving these problems: Chapters 2.3 in the script and/or Chapters 2.3 to 2.4 of *Peskin & Schroeder*.

Problem 1: *Commutation relations*
For a real scalar field $\phi(x)$ the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The canonical momentum density is $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$. The theory is quantized by promoting ϕ and π to operators in the Schrödinger picture with the commutation relations

$$\begin{aligned}[\phi(\vec{x}), \pi(\vec{y})] &= i\delta^{(3)}(\vec{x}-\vec{y}),\\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0.\end{aligned}$$

Introduce now the operators a and $a^\dagger$ by

$$\begin{aligned}\phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left\{a(\vec{p})\,e^{i\vec{p}\vec{x}} + a^\dagger(\vec{p})\,e^{-i\vec{p}\vec{x}}\right\},\\ \pi(\vec{x}) &= \frac{-i}{2}\int \frac{d^3p}{(2\pi)^3}\left\{a(\vec{p})\,e^{i\vec{p}\vec{y}} - a^\dagger(\vec{p})\,e^{-i\vec{p}\vec{x}}\right\}\end{aligned}$$

and derive the commutation relations

$$[a(\vec{p}), a^\dagger(\vec{q})], \quad [a(\vec{p}), a(\vec{q})], \quad [a^\dagger(\vec{p}), a^\dagger(\vec{q})].$$

Problem 2: *Complex scalar field: quantization*
Consider the Lagrangian density for a free complex scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi.$$

a) Show that the canonical momenta of ϕ and ϕ^* are

$$\pi = \dot{\phi}^*, \quad \pi^* = \dot{\phi}.$$

and derive an expression for the Hamiltonian H.

b) Proceed to quantization by promoting ϕ, ϕ^* and π, π^* to operators $\phi, \phi^\dagger$ and $\pi, \pi^\dagger$ (in the Schrödinger picture). What would you postulate as their commutation relations?

c) Introduce now creation and annihilation operators by writing

$$
\begin{aligned}
\phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ a(\vec{p})\, e^{i\vec{p}\vec{x}} + b^\dagger(\vec{p})\, e^{-i\vec{p}\vec{x}} \right\}, \\
\pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2} \left\{ -a^\dagger(\vec{p})\, e^{-i\vec{p}\vec{x}} + b(\vec{p})\, e^{i\vec{p}\vec{x}} \right\}.
\end{aligned}
$$

Why do we now need operators $b, b^\dagger$ in addition to $a, a^\dagger$? Convince yourself that the commutation relations

$$
\begin{aligned}
[a(\vec{p}), a^\dagger(\vec{q})] &= [b(\vec{p}), b^\dagger(\vec{q})] = 2\omega_{\vec{p}}\, (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}), \\
[a(\vec{p}), a(\vec{q})] &= [b(\vec{p}), b(\vec{q})] = 0, \\
[a(\vec{p}), b(\vec{q})] &= [a(\vec{p}), b^\dagger(\vec{q})] = 0
\end{aligned}
$$

are consistent with the ones postulated in part b).

d) Show that the Hamiltonian can be written as

$$
H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left\{ a^\dagger(\vec{p}) a(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p}) \right\} + \text{const.}
$$

Why is the positive sign in front of the $b^\dagger b$ term important? What is the physical interpretation of b and $b^\dagger$?

e) Switch now to the Heisenberg picture

$$
\phi_H(t, \vec{x}) = e^{iHt} \phi(\vec{x}) e^{-iHt}.
$$

Show that

$$
\begin{aligned}
e^{iHt}\, a(\vec{p})\, e^{-iHt} &= a(\vec{p})\, e^{-i\omega_{\vec{p}} t}, & e^{iHt}\, a^\dagger(\vec{p})\, e^{-iHt} &= a^\dagger(\vec{p})\, e^{i\omega_{\vec{p}} t}, \\
e^{iHt}\, b(\vec{p})\, e^{-iHt} &= b(\vec{p})\, e^{-i\omega_{\vec{p}} t}, & e^{iHt}\, b^\dagger(\vec{p})\, e^{-iHt} &= b^\dagger(\vec{p})\, e^{i\omega_{\vec{p}} t},
\end{aligned} \tag{1}
$$

and therefore

$$
\phi_H(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ a(\vec{p})\, e^{-ipx} + b^\dagger(\vec{p})\, e^{ipx} \right\}.
$$

PROBLEM 3: *"String quantization"*
Consider the Lagrangian of the clamped string derived in problem 1 on sheet 1

$$
\begin{aligned}
L &= \int_0^R dx \left\{ \frac{1}{2}\sigma \left(\frac{\partial y}{\partial t} \right)^2 - \frac{1}{2}\tau \left(\frac{\partial y}{\partial x} \right)^2 \right\} \\
&= \sum_{n=1}^{\infty} \left\{ \frac{1}{2}\, \sigma\, \dot{q}_n^2 - \frac{1}{2}\, \tau \left(\frac{n\pi}{R} \right)^2 q_n^2 \right\}.
\end{aligned}
$$

a) How would you quantize that string? What commutation relations would you postulate for the "field operator" $y(x)$ and its canonical momentum $(\pi_y)(x)$ or for the operators q_n and their canonical momenta $(\pi_q)_n$?

b) Express the Schrödinger picture operators q_n and its canonical momentum in terms of creation and annihilation operators $a_n, a_n^\dagger$. (Use a normalization as in the lecture.) Can you guess the form of the Hamiltonian H expressed in terms of these operators?