Problems for Quantum Field Theory 1

3. Sheet

Suggested reading before solving these problems: Chapter 2.3, 3.1 in the script and/or Chapters 2.2, 2.3 and 2.4 of *Peskin & Schroeder*.

Problem 1: *A commutation relation for the three-momentum operator*
Consider a real scalar field with Lagrangean density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

and conjugate momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi}$. You have learned how to quantize the system by promoting ϕ and π to operators with the commutation relations

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}), \qquad (1)$$

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0. \qquad (2)$$

The associated Hamiltonian and the three-momentum operators read

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{m^2}{2}\phi^2\right]$$

$$\vec{P} = \int d^3x \left(\pi\,\vec{\nabla}\phi\right)$$

Show that these operators commute: $[H, \vec{P}] = 0$ by *only* using the commutation relations (1) and (2).

Problem 2: *Charge of a complex scalar field*
Consider the Lagrangean density for a free complex scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi.$$

and define the associated conjugate momenta π and $\pi^\star$. The Noether theorem leads to a conserved charge, given in terms of 0-component of the Noether current:

$$Q \equiv \int d^3x\, j^0 .$$

For a complex scalar field, the four-vector associated with the current j reads

$$j^\mu = i\left[(\partial^\mu\phi)^\star\phi - \phi^\star(\partial^\mu\phi)\right],$$

from which an expression for the corresponding charge Q can be easily obtained.

The theory is quantised by promoting ϕ, ϕ^* and their conjugate momenta to operators. To that end it is convenient to introduce the creation and annihilation operators

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ a(\vec{p})\, e^{i\vec{p}\vec{x}} + b^\dagger(\vec{p})\, e^{-i\vec{p}\vec{x}} \right\}, \quad (3)$$

$$\pi(\vec{x}) = \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} \left\{ b(\vec{p})\, e^{i\vec{p}\vec{x}} - a^\dagger(\vec{p})\, e^{-i\vec{p}\vec{x}} \right\}. \quad (4)$$

with commutation relations

$$\begin{aligned} [a(\vec{p}), a^\dagger(\vec{q})] &= [b(\vec{p}), b^\dagger(\vec{q})] = 2\omega_{\vec{p}}\, (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}), \\ [a(\vec{p}), a(\vec{q})] &= [b(\vec{p}), b(\vec{q})] = 0, \\ [a(\vec{p}), b(\vec{q})] &= [a(\vec{p}), b^\dagger(\vec{q})] = 0\ . \end{aligned}$$

Express the charge Q in terms of the operators a, $a^\dagger$ and b, $b^\dagger$, carrying out all the details of the calculation.

PROBLEM 3: *Scalar field operator in Heisenberg picture*

In the Heisenberg picture, a real scalar field operator $\phi(\vec{x})$ can be made time dependent in the following way

$$\phi(t, \vec{x}) = e^{iHt} \phi(\vec{x}) e^{-iHt}$$

and analogously for its conjugate momentum. Consider now the expansion of $\phi(\vec{x})$ and $\pi(\vec{y})$ in terms of creation and annihilation operators, obtained by imposing reality conditions to equations (3) and (4). Show that the following relation holds for any n:

$$\vec{P}^n\, a(\vec{p}) = a(\vec{p}) \left(\vec{P} - \vec{p} \right)^n$$

where $\vec{P}$ is the three-momentum operator, with $\vec{p}$ the corresponding eigenvalue. Find the relation satisfied by the operator $a^\dagger(\vec{p})$. Deduce that the following identities hold

$$e^{-i\vec{P}\vec{x}} a(\vec{p}) e^{i\vec{P}\vec{x}} = a(\vec{p}) e^{i\vec{p}\vec{x}} \quad , \quad e^{-i\vec{P}\vec{x}} a^\dagger(\vec{p}) e^{i\vec{P}\vec{x}} = a^\dagger(\vec{p}) e^{-i\vec{p}\vec{x}} \quad .$$

With this information, show that the time dependent real scalar operator $\phi(t, \vec{x})$ can be expressed as

$$\phi(t, \vec{x}) = e^{i(Ht - \vec{P}\vec{x})}\, \phi(0, \vec{0})\, e^{-i(Ht - \vec{P}\vec{x})}\, .$$