Problems for Quantum Field Theory 1

4. Sheet

Suggested reading before solving these problems: Chapter 3 in the script and/or Chapter 4.2 of *Peskin & Schroeder.*

Problem 1: *Unitary evolution and T-product*

Consider the decomposition of a Hamiltonian operator H in free and interaction parts, $H = H_0 + H_{\rm int}$. In the interaction picture, operators evolve in time by means of the free Hamiltonian H_0, while states $|f\rangle$ evolve by means of the interaction Hamiltonian:

$$i\,\partial_t\,|f\rangle \;=\; H_{\rm int}(t)\,|f\rangle\,.$$

Show that this implies that we can write $|f(t)\rangle = U(t,\,t_0)\,|f(t_0)\rangle$, where the unitary operator $U(t, t_0)$ satisfies the differential equation (Schrödinger equation)

$$i\,\partial_t\,U(t,\,t_0) \;=\; H_{\rm int}(t)\,U(t,\,t_0)\,. \tag{1}$$

Show that the solution of equation (1) can be expressed as a power series, in which each term is an operator:

$$U(t,\,t_0) \;=\; \mathbf{1} - i\int_{t_0}^{t} dt_1 H_I(t_1) \,+\, (-i)^2\int_{t_0}^{t}\int_{t_0}^{t_1} dt_1 dt_2\, H_I(t_1)H_I(t_2) + \ldots \tag{2}$$

Convince yourself that the previous series can be re-expressed as

$$U(t,\,t_0) \;=\; T\left\{\exp\left[-i\int_{t_0}^{t} dt' H_I(t')\right]\right\}$$

where the T-product acts as $T\,A(t)B(t') \;=\; A(t)B(t')\,\Theta(t-t') + B(t')A(t)\,\Theta(t'-t)$. In particular, show that the expansion of the exponential up to second term provides eq. (2), and try to generalize your argument for the higher order terms.

Problem 2: *2 to 2 scattering*

In the lecture course you have learned how to describe the scattering of two particles in the interaction picture. Assume that the particles are characterised by momenta $\vec{p}_1$ and $\vec{p}_2$ in the initial state, and by momenta $\vec{p}_1^{\,\prime}$ and $\vec{p}_2^{\,\prime}$ in the final state. The interaction Hamiltonian is $H_I = \frac{\lambda}{4!}\,\phi^4$, where $\phi(t,\vec{x})$ is the time dependent operator associated with a real scalar field.

In particular, you have learned that the amplitude controlling the process can be obtained from the following quantity

$$i\,T_{fi} \;\simeq\; -i\,\langle 0|\,a(\vec{p}_1^{\,\prime})a(\vec{p}_2^{\,\prime})\,\left[\frac{\lambda}{4!}\int d^4x\,\phi^4(x)\right]\,a^\dagger(\vec{p}^{\,\prime})a^\dagger(\vec{p}_2)\,|0\rangle \tag{3}$$

by isolating the contributions proportional to $\delta^4(p_1+p_2-p_1'-p_2')$, and defining the scattering amplitude $\mathcal{M}_{fi}$ as

$$i\,T_{fi} \equiv i\,\mathcal{M}_{fi}\,(2\pi)^4\,\delta^4\left(p_1+p_2-p_1'-p_2'\right)$$

The amplitude can be extracted by expanding the scalar field $\phi(t,\,\vec{x})$ in terms of ladder operators $a(\vec{p})$ and $a^\dagger(\vec{p})$, and by plugging this expansion in eq. (3). Then, using the commutation relations of a and $a^\dagger$, one extracts the terms that are proportional to $\delta^4(p_1+p_2-p_1'-p_2')$.

- Identify the relevant terms in the expansion! How many are there? Do they all give the same contribution?
- Show that the final result is $\mathcal{M}_{fi} = -4!\,\frac{\lambda}{4!} = -\lambda$. Is there a connection between the coefficient in this result, and the number of terms in the expansion of eq. (3) that contribute to the scattering amplitude?
- Try to generalize the previous results to a theory with interaction Hamiltonian $H_I = \frac{\lambda}{n!}\,\phi^n$, with n being a natural number.
- By plugging the scalar field expansion in eq. (3), do you also find terms that are *not* proportional to $\delta^4(p_1+p_2-p_1'-p_2')$? If so, what is their physical interpretation?