

PROBLEMS FOR QUANTUM FIELD THEORY 1  
7. Sheet

Suggested reading before solving these problems: Chapters 4.1 in the script and/or Chapters 3.1 of *Peskin & Schroeder*.

PROBLEM 1: *Infinitesimal and finite Lorentz transformations*  
Under a Lorentz transformation the coordinates transform as

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

with a matrix  $\Lambda$  that leaves the metric tensor unchanged

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}.$$

- a) Show that an infinitesimal Lorentz transformation has the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

with  $\omega_{\mu\nu} = -\omega_{\nu\mu}$ .

- b) Write down the matrix  $\omega^\mu{}_\nu$  that correspond to a rotation through an infinitesimal angle  $\vartheta$  around the  $z$ -axis. Do the same for a boost along the  $z$ -axis by an infinitesimal velocity  $v$ .
- c) By exponentiating, deduce the form of  $\Lambda^\mu{}_\nu$  for a finite rotation around or a finite boost along the  $z$ -axis.

PROBLEM 2: *Generators of the Lorentz group.*

- a) Verify that the infinitesimal Lorentz transformations found in problem 1 can be written as

$$\Lambda = -\frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta}$$

where the matrix  $M^{\alpha\beta}$  has the components

$$(M^{\alpha\beta})^\mu{}_\nu = i(\eta^{\mu\alpha} \delta_\nu^\beta - \eta^{\mu\beta} \delta_\nu^\alpha).$$

- b) Show that the generators  $M^{\alpha\beta}$  of infinitesimal Lorentz transformations satisfy the commutation relations (Lie algebra brackets)

$$[M^{\alpha\beta}, M^{\gamma\delta}] = i(\eta^{\beta\gamma} M^{\alpha\delta} + \eta^{\alpha\delta} M^{\beta\gamma} - \eta^{\alpha\gamma} M^{\beta\delta} - \eta^{\beta\delta} M^{\alpha\gamma}).$$

- c) Show that

$$[M^{01}, M^{23}] = 0$$

and that this holds for all permutations of 0, 1, 2, 3.

d) The generators of rotations are

$$J_1 = M^{23}, \quad J_2 = M^{31}, \quad J_3 = M^{12}.$$

Check that the  $J_k$  satisfy the angular momentum commutation relation

$$[J_i, J_j] = i \epsilon_{ijk} J_k,$$

e) The generators of Lorentz boosts are

$$K_1 = M^{10}, \quad K_2 = M^{20}, \quad K_3 = M^{30}.$$

Check the commutation laws

$$[J_i, K_j] = i \epsilon_{ijk} K_k, \quad [K_i, K_j] = -i \epsilon_{ijk} J_k.$$

f) Show that the differential operators

$$\tilde{M}^{\alpha\beta} = i (x^\alpha \partial^\beta - x^\beta \partial^\alpha)$$

(with  $\partial_\mu = \frac{\partial}{\partial x^\mu}$ ) satisfy the commutation relations of the generators of Lorentz transformations in part b).

**PROBLEM 3: Transformation of fields.** A scalar field  $\phi(x)$  transforms under a Lorentz transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

according to

$$\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x).$$

In contrast, a vector field  $A^\mu(x)$  transforms as

$$A^\mu(x) \rightarrow A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x).$$

- What is the transformation behaviour of a tensor field  $T^{\mu\nu}(x)$  or more general  $T^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_m}(x)$ ?
- An example for a tensor field is the electromagnetic field strength  $F^{\mu\nu}$  while the current  $j^\mu$  is a vector field. Show that Maxwells equation

$$\partial_\mu F^{\mu\nu} = j^\nu$$

is invariant under Lorentz transformations.

- Show that the following special “field configurations” are invariant:

$$\phi(x) = \text{const.}, \quad A^\mu(x) = x^\mu, \quad T^{\mu\nu}(x) = \eta^{\mu\nu}, \quad T^\mu{}_\nu(x) = x^\mu x_\nu.$$