Problems for Quantum Field Theory 1

9. Sheet

Suggested reading before solving these problems: Chapter 4.2 and 4.3 in the script and/or Chapter 3.3 and 4.7 of *Peskin & Schroeder*.

Problem 1: *Properties of Dirac field*

Since a Dirac field satisfies a Klein-Gordon equation, it is natural to expand it in plane waves. In the lectures you have learned the properties of the quantities u_s and v_s, associated with positive and negative frequency waves respectively.

a) Prove the following relations,

$$\sum_{i=0}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m\,, \qquad \sum_{i=0}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m\,.$$

b) Show that

$$\bar{u}_r\,\gamma^0\,u_s(p) = 2\,p^0\,\delta_{rs}\,, \qquad \bar{v}_r\,\gamma^0\,v_s(p) = 2\,p^0\,\delta_{rs}\,.$$

c) Using *a*) and the expansion of ψ and $\bar{\psi}$ in terms of creation and annihilation operators, compute

$$\langle 0|\,\psi(x)\bar{\psi}(y)\,|0\rangle$$

d) Prove by using *c*) that the Feynman propagator is

$$\langle 0|\,T\,\psi(x)\bar{\psi}(y)\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{\not{p}+m}{p^2-m^2+i\epsilon}\,e^{-i\,p(x-y)}$$

e) Consider the conserved quantity $j^\mu = \bar{\psi}\gamma^\mu\psi$. Expanding ψ and $\bar{\psi}$, compute the corresponding charge

$$Q = \int d^3x\,j^0\,.$$

Problem 2: *Vacuum polarization*

Consider the following Lagrangean, describing a theory of interacting scalar and spinor fields:

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 - \frac{m_\phi^2}{2}\,\phi^2 + \bar{\psi}\,\left(i\not{\partial} - m_\psi\right)\psi - h\,\phi\,\bar{\psi}\psi\,.$$

a) Write in detail the Feynman rules associated with this theory.

b) Convince yourself that, due to the interaction with the fermion, the scalar propagator acquires a one-loop contribution. Draw the corresponding Feynman diagram.

c) Write the amplitude associated to the previous loop contribution. Show that it reads

$$-i\,\mathcal{M}^2(p^2) \;=\; -\,(-ih)^2 \int \frac{d^4q}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i\left(\not{q}+\not{p}+m_\psi\right)}{(q+p)^2-m_\psi^2}\,\frac{i\left(\not{q}+m_\psi\right)}{q^2-m_\psi^2}\right]$$

where p is the momentum of the incoming scalar field. Why there is an overall minus sign in the right hand side? And which is the role of the trace in the integrand?

d) Perform the trace by using $\mathrm{tr}\,\mathbb{1}=4$, $\mathrm{tr}\,\gamma_\mu=0$ and $\mathrm{tr}\,\gamma_\mu\gamma_\nu=2\eta_{\mu\nu}$. Convince yourself that the latter identity follows directly from the Clifford algebra. Plug the result in the integral, and show that

$$-i\,\mathcal{M}^2(p^2) \;=\; -\,4h^2 \int \frac{d^4q}{(2\pi)^4}\,\frac{q\,(p+q)+m_\psi^2}{\left[(q+p)^2-m_\psi^2\right]\left[q^2-m_\psi^2\right]}$$

e) Consider the so called Feynman trick:

$$\frac{1}{AB} \;=\; \int_0^1 dx dy\,\frac{\delta(1-x-y)}{(xA+yB)^2}$$

Use it to show that, after making an appropriate shift in q to remove the cross terms containing qp, our integral becomes

$$-i\,\mathcal{M}^2(p^2) \;=\; -\,4h^2 \int \frac{d^4q}{(2\pi)^4}\int_0^1 dx\,\frac{q^2-x(1-x)p^2+m_\psi^2}{\left[q^2+x(1-x)p^2-m_\psi^2\right]^2}$$

f) Now the angular integration can be performed straightforwardly. Show that, calling $\Delta \equiv x(1-x)p^2-m_\psi^2$, one can write

$$-i\,\mathcal{M}^2(p^2) \;=\; -\,\frac{8h^2}{(2\pi)^3}\left[\int_0^{+\infty} dq\int_0^1 dx\,\frac{q^5}{(q^2+\Delta)^2} - \int_0^{+\infty} dq\int_0^1 dx\,\frac{\Delta\,q^3}{(q^2+\Delta)^2}\right].$$

The last two integrals in q are divergent: use an ultraviolet cut-off Λ in order to regularize them. How does each integral diverge in the ultraviolet for large values of Λ?

Carry out the computation of $\mathcal{M}$ as far as you can (taking care with changing the order of the integrals in q and x).