# Problems for Quantum Field Theory 1

1. Tutorial

**Problem 1:** *Natural units*

In particle physics one often chooses units such that $\hbar = c = 1$ and measures energies in MeV. Complete the following translation table (The charge of a proton is $e = 1.6 \cdot 10^{-19}$ $C$, the vacuum permittivity is $\epsilon_0 = 8.85 \cdot 10^{-12}$ $C/(Vm)$):

| | SI units | Natural units |
|---|---|---|
| c | $3 \cdot 10^8$ m/s | 1 |
| $\hbar$ | $1.05 \cdot 10^{-34}$ Js | 1 |
| $m_e$ | $9.1 \cdot 10^{-31}$ kg | |
| $m_p$ | | 938.3 MeV |
| $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ | | |
| $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$ | $0.53 \cdot 10^{-10}$ m | |
| $G$ | $6.67 \cdot 10^{-11}$ $\mathrm{Nm}^2/(\mathrm{kg}^2)$ | |

**Problem 2:** *Upper and lower indices*

In special relativity one distinguishes between upper or contra-variant indices (as e.g. $x^\mu$) and lower or co-variant indices (e.g. $x_\mu$). The metric tensor

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is used to raise or lower indices:

$$x_\mu = \eta_{\mu\nu} x^\nu, \qquad x^\mu = \eta^{\mu\nu} x_\nu.$$

For $x^\mu = (x^0, \vec{x})$ and $p^\mu = (p^0, \vec{p})$ calculate

$$x_\mu, \qquad p \cdot x = p_\mu x^\mu$$

and show that

$$\eta^\mu{}_\nu = \delta^\mu{}_\nu, \qquad (\eta^{\mu\nu}) = (\eta_{\mu\nu}).$$

Problem 3: *Poincaré group*

Under a Poincaré transformation $(\Lambda, a)$ a coordinate vector $x^\mu$ transforms as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

Poincaré transformations leave the scalar product of differences of coordinate vectors, $(x-y)\cdot(x-y)$, invariant. Hence the matrix $\Lambda$ satisfies

$$\Lambda^\rho{}_\mu \, \Lambda^\sigma{}_\nu \, \eta^{\mu\nu} = \eta^{\rho\sigma}. \tag{1}$$

a) Show that the components of $\Lambda^{-1}$ are

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu.$$

b) Consider the product of two Poincaré transformations

$$(\Lambda, a) = (\Lambda_1, a_1)(\Lambda_2, a_2).$$

Determine $(\Lambda, a)$ and show that $\Lambda$ satisfies Eq. (1).

c) Determine the inverse transformation

$$(\Lambda, a)^{-1}.$$

*Remark:* The properties shown here together with associativity

$$\Big[(\Lambda_1, a_1)(\Lambda_2, a_2)\Big](\Lambda_3, a_3) = (\Lambda_1, a_1)\Big[(\Lambda_2, a_2)(\Lambda_3, a_3)\Big]$$

and the existence of a unit element

$$(\mathbb{1}, 0)$$

imply that the set of Poincaré transformations constitutes a *group*.