Problems for Quantum Field Theory 1

2. Tutorial

Problem 1: *Fourier transform and Dirac distribution*

Consider a three-dimensional cubus $x_i \in (-L/2, L/2), i = 1, 2, 3$ with volume $V = L^3$ and periodic boundary conditions. A (complex) field configuration $\phi(\vec{x})$ can be written as

$$\phi(\vec{x}) = \frac{1}{V} \sum_{l,m,n} e^{i\vec{p}_{lmn}\vec{x}}\, \tilde{\phi}_{lmn} \tag{1}$$

with $\vec{p}_{lmn} = \left(\frac{2\pi l}{L}, \frac{2\pi m}{L}, \frac{2\pi n}{L}\right)$.

a) What is the range of the indices l, m, n?

b) Show that

$$\int_V d^3x\, e^{i(\vec{p}_{lmn} - \vec{p}_{l'm'n'})\vec{x}} = V\, \delta_{ll'}\delta_{mm'}\delta_{nn'} \tag{2}$$

and hence

$$\phi_{lmn} = \int_V d^3x\, e^{-i\vec{p}_{lmn}\vec{x}}\, \phi(\vec{x}).$$

c) Consider now the limit $L \to \infty$. Show that Eq. (1) becomes

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\vec{x}}\, \tilde{\phi}(\vec{p}).$$

d) By comparing Eq. (2) to its infinite volume limit

$$\int d^3x\, e^{i(\vec{p}-\vec{p}')\vec{x}} = (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{p}')$$

convince yourself that

$$(2\pi)^3\, \delta^{(3)}(\vec{q})\big|_{\vec{q}=0} = \lim_{L\to\infty} V.$$