Quantum Field Theory II

Jan Martin Pawlowski, summer term 2010

Tuesday & Thursday, 9:15-11:00, gHS Pw 12 [LSF]

- Content
- Literature
- Exercises & bonus material
- Script
- QFT I

Prerequisites: quantum mechanics, classical field theory, statistics, basic knowledge of QFT

Content of lecture series

In the lecture course advanced topics in quantum field theory are discussed.

Outline

- **Path integral quantisation:**
  path integral, correlation functions, Feynman rules, lattice theory

- **Gauge theories:**
  Faddeev-Popov quantisation, BRST-symmetry, non-perturbative aspects

- **Renormalisation theory:**
  renormalisation group equations, beta-functions, renormalisability, fixed points & critical phenomena

- **Applications:**
  QCD, Operator product expansion (OPE), spontaneous symmetry breaking, standard model, anomalies & topology

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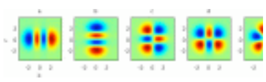


Literature

- ***Quantum field theory, basics***

Haag	Local Quantum Physics	Springer, 1996
Itzykson, Zuber	Quantum Field Theory	McGraw-Hill, 1980
Mandl, Shaw	Quantum Field Theory	Wiley, 1993
Peskin, Schroeder	An Introduction to Quantum Field Theory	Addison Wesley, 1995
Ramond	Field Theory. A Modern Primer	Addison Wesley, 1999
Ryder	Quantum Field Theory	Cambridge UP, 1996
Siegel	Fields	hep-th/9912205
Srednicki	Quantum Field Theory	Cambridge UP, 2007
Stone	The Physics of Quantum Fields	Springer, 2000
Weinberg	The Quantum Theory of Fields, Vol. 1-2	Cambridge UP, 1996

- ***Quantum field theory, applications***

Kugo	Eichtheorie	Springer, 1997
Miransky	Dynamical Symmetry Breaking in Quantum Field Theories	World Scientific, 1993
Muta	Foundations of Quantum Chromodynamics	World Scientific, 1987
Nachtmann	Elementarteilchenphysik - Phänomene und Konzepte	Vieweg,1992
Pokorski	Gauge Field Theories	Cambridge UP, 1987
Wu-Ki Tung	Group Theory in Physics	World Scientific, 1985
Zinn-Justin	Quantum Field Theory and Critical Phenomena	Oxford UP, 1993

- ***Textbooks on the renormalisation group and critical phenomena***

Amit	Field Theory, the	World Scientific

	Renormalization Group, and Critical Phenomena	
Binney, Dowrick, Fisher, Newman	The Theory of Critical Phenomena, an Introduction to the Renormalization Group	Clarendon Press, Oxford
Cardy	Scaling and Renormalization in Statistical Physics	Cambridge UP
Collins	Renormalization	Springer
Parisi	Statistical Field Theory	Addison-Wesley

- ***Textbooks on geometry & topology in physics***

Bertlmann	Anomalies in Quantum Field Theory	Oxford UP, 2000
Nakahara	Geometry, Topology and Physics	Hilger
Nash & Sen	Topology And Geometry For Physicists	Academic

1 Functional integral approach

1.1 Path integral in quantum mechanics

We aim at correlation fcts. & time evolution of QM systems: (1 dim)

$$\hat{H} = \hat{p}^2/2m + V(\hat{q}) \tag{1.1}$$

with $\hbar = c = 1$

$$[\hat{q}, \hat{p}] = i \quad \text{Heisenberg alg.} \tag{1.2}$$

Hilbert space $\mathcal{H}$ related to (1.2):

Space of square-integrable fcts.:

$$\Psi : q \to \Psi(q) \tag{1.3}$$

with

$$\int_{\mathbb{R}} dq\, |\Psi|^2(q) < \infty$$

Remark: The above is only a possible rep. of the Heisenberg alg. (1.1) and the Hilbert space.

Representation of ops.:

$$\hat{q}: \quad \hat{q}\,\psi(q) = q \cdot \psi(q) \qquad (1.4)$$

$$\hat{p}: \quad \hat{p}\,\psi(q) = -i \frac{\partial \psi}{\partial q}(q)$$

$$\hat{p} = -i \frac{\partial}{\partial q}$$

Eigen states:

$$\hat{p}\,|p\rangle = p\,|p\rangle \qquad (1.5)$$
$$\hat{q}\,|q\rangle = q\,|q\rangle$$

with continuous spectrum $q, p \in \mathbb{R}$.

Normalisation: $\langle q | q' \rangle = 0$ for $q \neq q'$

$\int_{\mathbb{R}} dq' \, |q'\rangle\langle q'| = 1 \quad \Longrightarrow \quad \int_{\mathbb{R}} dq' \, \langle q | q' \rangle \langle q' | = \langle q |$

$$\Rightarrow \langle q | q' \rangle = \delta(q - q') \qquad (1.6)$$

We also define

$$\langle p | p' \rangle = 2\pi\, \delta(p - p') \qquad (1.7)$$

Hence

$$\langle p | q \rangle = e^{-ipq} \qquad (1.8)$$

Correlation fcts. : transition amplitude
from initial state $|q_{in}\rangle$ at t_0
to final state $|q_f\rangle$ at $t_f = t$

see QFT I, chapter 3

time evolution op.:

$$\boxed{i\,\partial_t\, \mathcal{U}(t,t_0) = \hat{H}(t)\,\mathcal{U}(t,t_0)} \qquad (1.9)$$

$\Rightarrow$ transition amplitude: $\Delta t = \frac{t-t_0}{n}$

$$\langle q_f|\mathcal{U}(t,t_0)|q_{in}\rangle$$

$$= \langle q_f|\mathcal{U}(t,t-\Delta t)\cdots\mathcal{U}(t-(n-1)\Delta t, t_0)|q_{in}\rangle$$

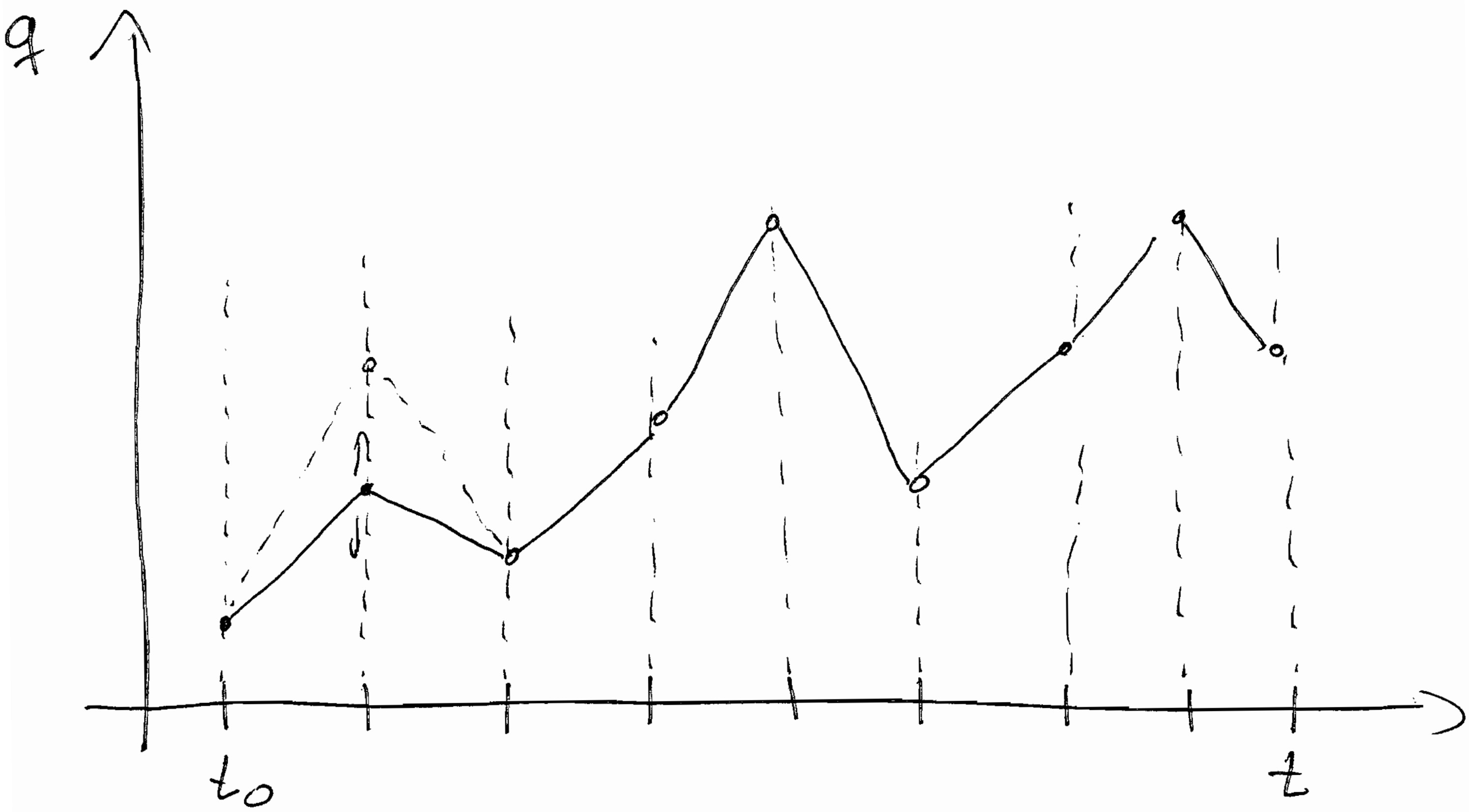
$$= \left[\prod_{i=1}^{n-1}\int dq_i\right]\langle q_f|\mathcal{U}(t,t-\Delta t)|q_{n-1}\rangle\langle q_{n-1}|\mathcal{U}(t-\Delta t, t-2\Delta t)|q_{n-2}\rangle$$

$$\cdots\langle q_1|\mathcal{U}(t-(n-1)\Delta t, t_0)|q_{in}\rangle \qquad (1.10)$$

where we have used

$$\mathcal{U}(t,t_1)\,\mathcal{U}(t_1,t_2) = \mathcal{U}(t,t_2)$$

$$\text{for } t \geq t_1 \geq t_2 \qquad (1.11)$$



$$q_i = q(t_i)$$

For $\Delta t \to 0$: $\quad t_i = t_0 + i\,\Delta t \quad , t = t_n$

$$\mathcal{U}(t_i, t_i - \Delta t) = \underbrace{1 - i\,\hat{H}\,\Delta t}_{\sim\, e^{-i\hat{H}\Delta t}} + O(\Delta t^2) \qquad (1.12)$$

satisfies (1.9).

Hence

$$\langle q_i | \mathcal{U}(t_i, t_{i-1}) | q_{i-1} \rangle$$

$$\simeq \langle q_i | (1 - i\hat{H}\Delta t) | q_{i-1} \rangle$$

$$= \int \frac{dp}{2\pi} \langle q_i | p \rangle \left(\langle p | q_{i-1} \rangle - i \langle p | \hat{H} | q_{i-1} \rangle \Delta t \right)$$

$$= \int \frac{dp}{(2\pi)}\, e^{+ipq_i} \left(e^{-ipq_{i-1}} - i \langle p | \hat{H} | q_{i-1} \rangle \Delta t \right)$$

$$(1.13)$$

We use that: (eq. (1.1))

$$\langle p | \hat{H} | q \rangle = \langle p | \hat{p}^2/2m + V(\hat{q}) | q \rangle$$

$$= \left(p^2/2m + V(q) \right) \langle p | q \rangle$$

$$= \left(p^2/2m + V(q) \right) e^{-ipq} \qquad (1.14)$$

Remark: in eq. (1.14) we have used that $\hat{H} = \hat{H}_1(\hat{p}) + \hat{H}_2(\hat{q})$. For general $\hat{H}$ $\langle p | \hat{H} | q \rangle = H(p,q) \langle p | q \rangle$ does <u>not</u> hold.

We conclude

$$\langle q_i | \mathcal{U}(t_i, t_{i-1}) | q_{i-1} \rangle$$

$V(q_{i-1}) = V(q_i) + O(\Delta t)$

$$\simeq \int \frac{dp}{(2\pi)} e^{ip(q_i - q_{i-1})} \left[1 - i \left(p^2/2m + V(q_i) \right) \Delta t \right]$$

$$\simeq \int \frac{dp}{(2\pi)} e^{i\{ p(q_i - q_{i-1}) - \left(p^2/2m + V(q_i) \right) \Delta t \}} \tag{1.15}$$

$$q_i - q_{i-1} = \frac{q_i - q_{i-1}}{\Delta t} \cdot \Delta t$$

discrete derivative $\dot{q}_i$

It follows:

$(2\pi)^n$

$$\langle q_f | \mathcal{U}(t, t_0) | q_{in} \rangle \simeq \int \left[\prod_i dq_i \, dp_i \right] \cdot e^{i \Delta t \sum_i \left(\dot{q}_i p_i - H(p_i, q_i) \right)} \tag{1.16}$$

with $H(p, q) = p^2/2m + V(q)$. Finally, $\Delta t \to 0$:

$$\boxed{\langle q_f | \mathcal{U}(t, t_0) | q_{in} \rangle \simeq \int \mathcal{D}q \, \mathcal{D}p \; e^{i \int_{t_0}^{t} dt' \left(\dot{q}(t) p(t) - H(p, q) \right)}} \tag{1.17}$$

with
$$\mathcal{D}q = \prod_i dq_i = \prod_i dq(t_i) \Big|_{\substack{q(t_0)=q_{in} \\ q(t)=q_f}} \qquad (1.18)$$
$$\mathcal{D}p = \prod_i dp_i = \prod_i dp(t_i)$$

Now we use that

$$\int_{\mathbb{R}} \frac{dp}{(2\pi)} e^{i\left(p\dot{q} - p^2/2m\right)} = \int \frac{dp}{(2\pi)} e^{-i(p-\dot{q}\cdot m)^2/2m} \cdot e^{i\dot{q}^2 m/2}$$

$$= \sqrt{\frac{m}{2\pi}} \cdot e^{i m \dot{q}^2/2} \qquad (1.19)$$

and arrive at

$$\boxed{\langle q_f | \mathcal{U}(t,t_0) | q_{in} \rangle = \frac{1}{N} \int \mathcal{D}q \; e^{iS[q]}} \qquad (1.20)$$

with $S[q] = \int_{t_0}^{t} dt' \Big\{ \underbrace{\tfrac{1}{2} m \dot{q}^2 - V(q)}_{L(q,\dot{q})} \Big\}$

N: Normalisation, see eq. (1.16). It will be taken care of later.

Correlation functions

We repeat the analysis in the presence of further position operators : $t \geq t_n \geq \ldots \geq t_1 \geq t_0$

$$\langle q_f | \mathcal{U}(t,t_n)\, \hat{q}\, \mathcal{U}(t_n,t_{n-1}) \ldots \mathcal{U}(t_2,t_1)\, \hat{q}\, \mathcal{U}(t_1,t_0) | q_{in} \rangle \tag{1.21}$$

$$= \langle q_f, t |\ \hat{q}(t_n) \ldots \hat{q}(t_1) | q_{in}, t_0 \rangle$$

Heisenberg pic.

with $\hat{q}(t) = \mathcal{U}(0,t)\, \hat{q}\, \mathcal{U}(t,0)$

$|q,t\rangle = \mathcal{U}(0,t)\, |q\rangle$

$$n=0 : \quad \langle q_f, t | q_{in}, t_0 \rangle \simeq \int \mathcal{D}q\, e^{iS[q]} \tag{1.22}$$

$n=1$: $\langle q_f, t | \hat{q}(t_1) | q_{in}, t_0 \rangle$

$$= \int dq \langle q_f, t | q, t_1 \rangle\ q\ \langle q, t_1 | q_{in}, t_0 \rangle$$

$$= \int dq\ q \quad \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_1)=q}} e^{iS[q]} \int \mathcal{D}q \Big|_{\substack{q(t_1)=q \\ q(t_0)=q_{in}}} e^{iS[q]}$$

$$= \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_0)=q_{in}}} q(t_1)\, e^{iS[q]} \tag{1.23}$$

in general: $t \geq t_n \geq \cdots \geq t_1 \geq t_0$

$$\langle q_f, t | \hat{q}(t_n) \cdots \hat{q}(t_1) | q_{in}, t_0 \rangle$$

$$= \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_0)=q_{in}}} q(t_n) \cdots q(t_1)\, e^{iS[q]} \qquad (1.24)$$

In most cases we are interested in vacuum amplitudes (see also QFT I, chapter)

$$\langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \qquad (1.25)$$

For the projection on the vacuum $|0\rangle$, $H|0\rangle = E_0|0\rangle$ the lowest energy state, we introduce a damping factor: (eq. (1.16), (1.17))

$$e^{-i\Delta t H} \to e^{-i\Delta t(1-i\varepsilon)H}$$

$$= e^{-i\Delta t H}\, e^{-\Delta t H \varepsilon} \qquad (1.26)$$

Higher energy states are suppressed by $e^{-(E-E_0)\Delta t \varepsilon}$ at each time step.

It follows that

$$\lim_{T\to\infty} \langle q_f, T | \hat{q}(t_n) \cdots \hat{q}(t_1) | q_{in}, -T \rangle$$
$$\simeq \langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \qquad (1.27)$$

Remark: Eq. (1.27) holds for all states which overlap with the vacuum

$$\langle q_f, T | 0 \rangle \neq 0 .$$

Hence we have: $t_n \geq t_{n-1} \geq \cdots \geq t_1$

$$\langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle$$
$$= \int \mathcal{D}q \; q(t_n) \cdots q(t_1) \, e^{i S_\varepsilon[q]} \qquad (1.28)$$

with

$$S_\varepsilon[q] = \int dt \left(\frac{1}{2} m \dot{q}^2 - V(q) + i\varepsilon q^2 \right) \qquad (1.29)$$

Eq. (1.29) has the same effect as eq. (1.26).

The derivation of eq. (1.28) was done with $t_n \geq t_{n-1} \geq \cdots \geq t_1$. The path integral or functional integral on the rhs of eq. (1.28) can be written down for arbitrary times $t_1, \ldots, t_n$. Hence we have finally

$$\boxed{\begin{aligned} &\langle 0 | T\, \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \\ &\quad \simeq \int \mathcal{D}q \; q(t_n) \cdots q(t_1)\, e^{iS[q]} \end{aligned}} \tag{1.30}$$

where the ε-prescription eq. (1.29) is understood. Normalised correlation functions read

$$\boxed{\frac{\langle 0 | T\, \hat{q}(t_1) \cdots \hat{q}(t_n) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\int \mathcal{D}q \; q(t_1) \cdots q(t_n)\, e^{iS[q]}}{\int \mathcal{D}q \; e^{iS[q]}}} \tag{1.31}$$

Remarks:

(1) The quadratic part of the action S_ε reads ($m=1$)

$$S_\varepsilon[q] = \frac{1}{2}\int dt\; q(t)\left[-\partial_t^2 - \omega^2 + i\varepsilon\right] q(t) \qquad (1.32)$$

The propagator that follows from eq. (1.32) is in momentum space (frequency space)

$$\frac{1}{p^2 - \omega^2 + i\varepsilon} = C \qquad (1.33)$$

covariance

wich is the time-ordered (Feynman) propagator.

(2)* Changing S_ε in eq. (1.32), eq. (1.33) to path integrals that do <u>not</u> describe time-ordered products.

$$\boxed{\text{Choice of covariance } C \iff \text{choice of ordering}}$$

Generating Functional

The expectation values $\frac{\langle 0|T\hat{q}(t_1)\cdots\hat{q}(t_n)|0\rangle}{\langle 0|0\rangle}$

are the normalised moments of the path integral $\int \mathcal{D}q\, e^{iS[q]}$.

Toy example: consider $\int_{\mathbb{R}} dq\, e^{iS(q)}$ with moments

$$\langle q^n\rangle := \frac{1}{N}\int_{\mathbb{R}} dq\; q^n\, e^{iS(q)} \qquad (1.34)$$

$\int dq\, e^{iS(q)}$

These moments are generated by

$$Z(j) = \frac{1}{N}\int dq\, e^{i\{S(q)+j\cdot q\}} \qquad (1.35)$$

$$\Rightarrow \quad \langle q^n\rangle = (-1)^n \frac{\partial^n Z}{\partial j^n}$$

Gaußian theory: $S_0(q) = \frac{1}{2} q \alpha q$

$$\Rightarrow Z_0(j) = \frac{1}{N_0} \int_{\mathbb{R}} dq \, e^{i\{\frac{1}{2} q \alpha q + j q\}}$$

$$= \frac{1}{N_0} \int_{\mathbb{R}} dq \, e^{i\{\frac{1}{2}(q + \frac{1}{\alpha} j)^2 - \frac{1}{2} j \frac{1}{\alpha} j\}}$$

$$= e^{-i/2 \, j \boxed{\frac{1}{\alpha}} j}$$

Covariance / Propagator

Moments:

$$\langle q^{2n} \rangle = \frac{2n!}{n!} \frac{1}{2^n} \frac{1}{\alpha^n}$$

Interacting theory: $S(q) = \frac{1}{2} q \alpha q - V(q)$

$$Z(j) = \frac{N_0}{N} e^{-i V(-i \frac{\partial}{\partial j})} Z_0(j)$$

Perturbation theory: expansion in powers of V

See exercise

QM: $q_{} \to q(t)$
$j \to j(t)$: derivative $\to$ functional derivative

$$\boxed{\frac{\delta j(t)}{\delta j(t')} := \delta(t-t')} \qquad (1.36)$$

It follows that

$$\frac{\delta}{\delta j(t_1)} e^{i\int_{\mathbb{R}} dt\, j(t) q(t)} = i \int_{\mathbb{R}} dt \underbrace{\frac{\delta j(t)}{\delta j(t_1)}}^{\delta(t-t_1)} q(t)\, e^{i\int_{\mathbb{R}} \cdots}$$

$$= i\, q(t_1)\, e^{i\int_{\mathbb{R}} dt\, j(t) q(t)} \qquad (1.37)$$

$\Rightarrow$ Generating functional:

$$Z[j] = \frac{1}{N} \int \mathcal{D}q\, e^{i\{S[q] + \int dt\, j(t) q(t)\}} \qquad (1.38)$$

with $N = \int \mathcal{D}q\, e^{iS[q]}$

with

$$\langle 0 | T \hat{q}(t_1) \cdots \hat{q}(t_n) | 0 \rangle / \langle 0 | 0 \rangle$$

$$= (-i)^n \frac{\delta^n Z[j]}{\delta j(t_1) \cdots \delta j(t_n)} \bigg|_{j=0} \qquad (1.39)$$

$F[j]$ functional of $j(t)$, e.g. $F[j]=\int_{\mathbb{R}} dt\; j(t)\, q(t)$

functional derivative: $\mathcal{D}_g F[j]$

$$F[j+\varepsilon g] = F[j] + \mathcal{D}_g F[j]\,\varepsilon + O(\varepsilon^2)$$

With $F[j] = \int_{\mathbb{R}} dt\; j(t)\, q(t)$

$$\Rightarrow \mathcal{D}_g F[j] = \int_{\mathbb{R}} dt\; g(t)\, q(t)$$

Eq. (1.36): $\mathcal{D}_g F[j] = q(t') \Rightarrow g_{t'}(t) = \delta(t-t')$

Also:
$$\mathcal{D}_g F[j] = \frac{\partial F[j+\varepsilon g]}{\partial \varepsilon}$$

Gaußian theory: $S_0[q] = \frac{1}{2}\int dt\, q(t)\left(-\partial_t^2 - \overset{V''[0]}{\omega^2}\right)q(t)$

see p. 11

Generating functional:

$$Z_0[j] = \frac{1}{N_0}\int \mathcal{D}q\, e^{i\{S[q] + \int dt\, j(t)\, q(t)\}}$$

$$= \frac{1}{N_0}\int \mathcal{D}q\, e^{iS[q']}$$

$$\cdot\, e^{-\frac{1}{2}\int dt\, dt'\, q(t)\, G(t,t')\, q(t')} \qquad (1.40)$$

with

$$q'(t) = q(t) - i\int dt'\, G(t,t')\, j(t')$$

$$(-\partial_t^2 - \omega^2)\, G(t,t') = i\delta(t-t') \qquad (1.41)$$

The path integral measure is translationinvariant

$$\mathcal{D}q = \mathcal{D}q' \qquad (1.42)$$

which follows directly from the translation invariance of $dq = dq'$ and

$\mathcal{D}q = \prod_i dq(t_i)$, eq. (1.18), p. 6

Consequently we have

$$\boxed{Z_0[j] = e^{-\frac{1}{2}\int dt\,dt'\,q(t)\,G(t,t')\,q(t')}} \quad (1.43)$$

with the propagator $G(t,t')$. Note that the choice of $G(t,t')$ determines the path integral.

Interacting theory: $S[q] = S_0[q] - \int dt\, V(q)$

$$\boxed{Z[j] = \frac{N_0}{N}\, e^{-i\int dt\, V\left(-i\frac{\delta}{\delta j}\right)}\, Z_0[j]} \quad (1.44)$$

with moments

$$\boxed{\begin{aligned} &\langle 0|T\hat{q}(t_1)\cdots\hat{q}(t_n)|0\rangle / \langle 0|0\rangle \\ &\qquad = \left.\frac{\delta^n Z[j]}{\delta\hat{q}(t_1)\cdots\delta\hat{q}(t_n)}\right|_{j=0} \end{aligned}} \quad (1.45)$$