

1.2 Functional integral for scalar fields

Consider now a scalar field in d dim.

$$\varphi = \varphi(x) \quad \text{with } x \in \mathbb{R}^d \quad (1.46)$$

with $d-1$ spatial dimensions, $x = (x_0, \vec{x})$.

Action:

$$S[\varphi] = \int d^d x \underbrace{\left\{ \frac{1}{2} \partial_\nu \varphi \partial^\nu \varphi - V(\varphi) \right\}}_{\mathcal{L} : \text{Lagrange density}}$$

$$\text{Hamiltonian: } H = \int d^{d-1} x \left\{ \underbrace{\pi(\vec{x})}_{\parallel \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}} \partial_0 \varphi - \mathcal{L} \right\} \quad (1.47)$$

$$= \int d^{d-1} x \left\{ \frac{1}{2} \pi_0 \dot{\varphi}^0 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + V(\varphi) \right\}$$

As in QM we consider the transition amplitude

$$\langle \varphi_f | \mathcal{U}(t, t_0) | \varphi_{in} \rangle \quad (1.48)$$

↑
final state

↑
in state

Completeness relation:

$$\mathbb{1} = \int \mathcal{D}\varphi(z) |\varphi\rangle\langle\varphi|, \quad \langle\varphi|\varphi'\rangle = \delta[\varphi(z) - \varphi'(z)] \quad (1.51)$$

Repeating all steps in QM in the present field theoretical setting leads to

$$\begin{aligned} & \langle 0|T \hat{\varphi}(x_1) \dots \hat{\varphi}(x_n)|0\rangle / \langle 0|0\rangle \\ &= \frac{\int \mathcal{D}\varphi \varphi(x_1) \dots \varphi(x_n) e^{iS[\varphi]}}{\int \mathcal{D}\varphi e^{iS[\varphi]}} \end{aligned} \quad (1.52)$$

with $S = \underbrace{\int d^d x \left\{ \frac{1}{2} \varphi(x) \left[-\Delta - m^2 + i\epsilon \right] \varphi(x) \right\}}_{S_0} - V_{\text{int}}(\varphi)$ time-ordering (1.53)

and

$$\mathcal{D}\varphi = \prod_{\vec{i} \in \text{lattice}} d\varphi_{\vec{i}} \quad (1.54)$$

$$\vec{i} = (i_0, \vec{i})$$

Alternative representations:

$$\varphi(x) = \sum_n c_n \varphi_n(x) \quad (1.54)$$

with $(\varphi_n, \varphi_m) = \delta_{nm}$ $\{\varphi_n\}$ orthonormal system
 $\int d^d x \varphi_n(x) \varphi_m(x)$

$$\Rightarrow \boxed{\mathcal{D}\varphi \simeq \prod_n d c_n} \quad (1.55)$$

Hence $\int \mathcal{D}\varphi e^{iS[\varphi]} \simeq \int \prod_n d c_n e^{iS[\varphi]}$

Example: free theory (Gaussian)

$$S[\varphi] = S_0[\varphi] \quad \leftarrow \text{eq. (1.53)} \quad (1.56)$$

Choice of φ_n for (1.55):

$$\boxed{(-\Delta - m^2) \varphi_n = \lambda_n \varphi_n} \quad (1.57)$$

with $\lambda_n \in \mathbb{R}^+$.

It follows:

$$\int \mathcal{D}\varphi e^{iS_0[\varphi]} \approx \int \prod_n d c_n e^{i/2 \sum_n c_n^2 \lambda_n} \quad (1.58)$$

with $S_0[\varphi] = i/2 \sum_{n,m} \int d^d x c_n c_m \cdot \underbrace{\varphi_n(x) [-\Delta - m^2 + i\epsilon] \varphi_m(x)}_{\delta_{nm}}$
 drop $i\epsilon \rightarrow$

$$= i/2 \sum_{n,m} c_n c_m \lambda_n \int d^d x \varphi_n(x) \varphi_m(x)$$

$$= i/2 \sum_n c_n^2 \lambda_n \quad (1.59)$$

Finally:

$$\int \mathcal{D}\varphi e^{iS_0[\varphi]} \approx \prod_n \lambda_n^{-1/2} = \det^{-1/2}(-\Delta - m^2) \quad (1.60)$$

The functional integral of the free theory gives us the $\det^{-1/2}$ of the kinetic operator (in the free theory).