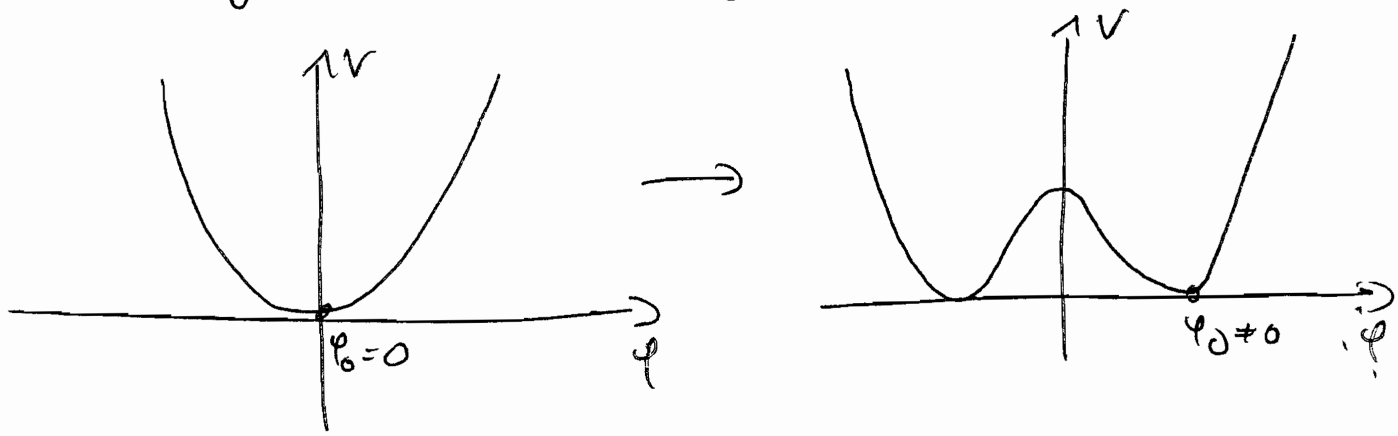
1.4 Effective action & spontaneous symmetry breaking

In the first lecture course QFT I we have briefly discussed spontaneous symmetry breaking, e.g.



Here, we want to use the functional integral for computing the effective potential at one loop: $V_{classical} \to V_{eff}$.

Consider the generating functional $Z[J]$,

$$Z[J] \simeq \int d\varphi \, e^{-S[\varphi] + \int d^d x \, J(x) \varphi(x)} \qquad (1.93)$$

with J such that

$$\boxed{\phi := \langle \varphi \rangle_J = \frac{\delta}{\delta J} W[J] \Big|_{J_\phi}} \qquad (1.94)$$

Now we compute $Z[J]$ in a <u>saddle point expansion</u> about the minimum ϕ the exponent in eq. (1.93),

$$\left.\frac{\delta S}{\delta \varphi}\right|_{\phi} = J \tag{1.95}$$

and hence

$$S[\varphi] + \int_x J\cdot\varphi = S[\phi] + \int_x J\phi + \frac{1}{2}\int_{x,y} \varphi\cdot S^{(2)}[\phi]\cdot\varphi + O(\varphi^3) \tag{1.96}$$

with

$$S^{(n)}[\phi](x_1,\ldots,x_n) = \frac{\delta^n S[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)} \tag{1.97}$$

Dropping the higher order terms in eq. (1.99), we get

$$\Gamma_{1\text{-loop}}[\phi] := \underbrace{\int_x J\cdot\phi - W}_{\text{is Legendre trafo of}} \simeq -\ln\int d\varphi\, e^{-\frac{1}{2}\int_{x,y}\varphi\cdot S^{(2)}[\phi]\cdot\varphi} + S[\phi] \tag{1.98}$$

which gives us, by using eq. (1.80) with $(-\Delta + m^2) \to S^{(2)}[\phi]$:

$$\boxed{\Gamma_{1\text{-loop}}[\phi] = S[\phi] + \frac{1}{2}\,\mathrm{Tr}\ln S^{(2)}[\phi]} \tag{1.99}$$

Example: φ^4-theory in 3d/4d (Euclidean)

$$S[\varphi] = \int d^dx \left\{ \frac{1}{2}\varphi\cdot(-\Delta + m_0^2)\cdot\varphi + \underbrace{\frac{\lambda_0}{4!}\varphi^4}_{V_0(\phi)} \right\} \qquad (1.100)$$

and
$$S^{(2)}[\phi](x,y) = \Big(-\Delta + \overbrace{m_0^2 + \frac{\lambda_0}{2}\phi^2}^{V_0''(\phi)}\Big)\,\delta^d(x-y)$$

For constant ϕ we have: $S^{(2)}[\phi](p,q) = S^{(2)}[\phi](p)(2\pi)^d\delta^d(p-q)$

$$S^{(2)}[\phi](p) = p^2 + m_0^2 + \frac{\lambda_0}{2}\phi^2 \qquad (1.101)$$

This gives us for the (1-loop) effective pot. $\boxed{V_{\rm eff} = \frac{1}{\underbrace{\int d^dx}_{\rm Vol.}}\Gamma[\phi]\Big|_{\phi_c}}$

$$\boxed{V_{\rm eff}(\phi) = V_0(\phi) + \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\ \ln\Big(p^2 + m_0^2 + \frac{\lambda_0}{2}\phi^2\Big) + \ln N}$$

(1.102)

We compute $V_{\rm eff}$ by taking its m_0^2-derivative:

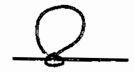
$$\partial_{m_0^2}V_{\rm eff}(\phi, m_0^2) = \partial_{m_0^2}V_0(\phi) + \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\left[\frac{1}{p^2+m_0^2+\frac{\lambda_0}{2}\phi^2} - \frac{1}{p^2+m_0^2}\right] \qquad (1.103)$$

Choose $\partial_{m_0^2} N = -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m_0^2}$

Eq. (1.103) is the one-loop approx. to the (unrenormalized) Callan-Symanzik equation, that governs $\partial_{m_0^2}\Gamma$.

In 3d, eq. (1.103) is finite. Upon momentum integration we get

$$\partial_{m_0^2} V_{eff}(\phi; m_0^2) = \partial_{m_0^2} V_0(\phi) - \frac{1}{8\pi}\left\{ \sqrt{m_0^2 + \tfrac{\lambda}{2}\phi^2} - \sqrt{m_0^2} \right\} \qquad (1.104)$$

This finiteness relates to the fact that $\partial_{m_0^2}$  is finite in 3d.

Using $\partial_{\phi^2} V_{eff} = \frac{\lambda}{2} \partial_{m_0^2} V_{eff}$, we have $V_{eff} \simeq \frac{\lambda}{2} \int d\phi^2\, \partial_{m_0^2} V_{eff}$ and hence

λ does not get renormalised in 3d

$$\boxed{V_{eff}(\phi) \simeq V(\phi) - \frac{1}{12\pi}\left(m^2 + \frac{\lambda}{2}\phi^2\right)^{3/2} + \lambda m \phi^2 \text{ - correct.}} \qquad (1.105)$$

renormalised

Remark: The nth field derivatives of $V_{eff}(\phi)$ at $\phi = 0$ give the amputated n-point Green functions at one loop.

(Exercise: Prove this for $n = 2, 4$)

In $d=4$, eq. (1.103) is not finite. We can either take a further m_0^2-derivative, or we use renormalisation: $\left[N = -\frac{1}{2}\int d^d p \ln(p^2+m^2)\right]$

$$m_0^2 = m^2 - \lambda/2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2}$$

$$\lambda_0 = \lambda + \frac{3}{2}\lambda^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^2} \qquad (1.106)$$

(RG-scale $p^2 = m^2$)

Inserting eq. (1.106) in eq. (1.102) we get

$$\boxed{V_{eff}(\phi) = \underbrace{V(\phi)}_{\frac{1}{2} m \phi^2 + \lambda/4!\, \phi^4} + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\left\{ \ln \frac{p^2+m^2+\lambda/2\,\phi^2}{p^2+m^2} - \frac{\lambda}{2}\frac{1}{p^2+m^2}\frac{\phi^2}{2} + (\lambda/2\,\phi^2)^2 \frac{1}{(p^2+m^2)^2}\right\}} \qquad (1.107)$$

Upon momentum integration this yields

$$\boxed{V_{eff}(\phi) \simeq \frac{1}{64\pi^2}(m^2 + \lambda/2\,\phi^2)^2 \ln(m^2 + \lambda/2\,\phi^2) \qquad (1.108)}$$