4 Non-Abelian gauge theories

(see also chapter 8.1, QFT I)

4.1 Action & gauge invariance

Consider fermions that carry some (matter: fundamental) representation of a non-Abelian group G, in most cases $G = SU(N)$.

$$\psi(x) \to \psi^{\mathcal{U}}(x) = \mathcal{U}\,\psi(x) \qquad \text{with } \mathcal{U} \in SU(N) \tag{4.1}$$

that is

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \quad \psi_i \text{ Dirac fermion}$$

in fundamental rep.

Dirac action invariant under <u>local</u> transformations (4.1): $\mathcal{U} = \mathcal{U}(x)$

$$S_D[\psi,\bar\psi,A] = -\int d^dx\ \bar\psi(x)\left(\not{D} + m\right)\psi(x) \tag{4.2}$$

with

$$D_\mu = \partial_\mu + i g A_\mu \tag{4.3}$$

covariant derivative

S_D in eq. (4.2) is invariant under local transformations, if

$$D_\mu \to \mathcal{U}(x) \mathcal{D}_\mu \mathcal{U}^\dagger(x) \qquad (4.4)$$

which entails

$$g A_\mu \to g A_\mu^{\mathcal{U}} = \mathcal{U}(x) g A_\mu(x) \mathcal{U}^\dagger(x) - i \mathcal{U}(x) \partial_\mu \mathcal{U}^\dagger(x)$$
$$= -i \, \mathcal{U}(x) \mathcal{D}_\mu \mathcal{U}^\dagger(x) \qquad (4.5)$$

A_μ has to live in the Lie algebra of $SU(N)$ [$su(N)$]

$\Rightarrow A_\mu$ matrix-valued

We write $\underset{\text{group}}{\mathcal{U}} = e^{i \overset{\text{algebra}}{\omega}}$ (4.6)

and infinitesimally

$$g A_\mu \to g A_\mu + i g [\omega(x), A_\mu] - \partial_\mu \omega + O(\omega^2)$$
$$= g A_\mu - \mathcal{D}_\mu \omega + O(\omega^2) \qquad (4.7)$$

adjoint represent.

with $\mathcal{D}_\mu \omega = \partial_\mu \omega + i g [A_\mu, \omega]$ (4.8)

Invariance of S_D in eq. (4.2):

$$S_D[\psi^U, \bar\psi^U, A_\mu^U] = -\int d^dx\, \bar\psi(x)\, \underbrace{U^\dagger(x)\, U(x)}_{\mathbb{1}}\, (\not{D} + m)\, U^\dagger(x) U(x) \cdot \psi(x)$$

$\bar\psi^U = \bar\psi U^\dagger$ $\qquad$ $U \in SU(N)$

$$= S_D[\psi, \bar\psi, A_\mu] \qquad (4.9)$$

Gauge-invariant action for gauge field:

Field strength:

$$F_{\mu\nu} = \frac{1}{gi}[D_\mu, D_\nu] \qquad (4.10)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

with

$$F_{\mu\nu} = F_{\mu\nu}^a\, t^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$$

generators of $SU(N)$

(4.11)

$$A_\mu = A_\mu^a\, t^a$$

The generators t^a satisfy the Lie-algebra

$$\boxed{[t^a, t^b] = i f^{abc} t^c} \qquad (4.12)$$

t^a selfadjoint $\qquad$ structure constants

Evidently, the field strength $F_{\mu\nu}$ transforms as a <u>tensor</u> under gauge transformations

$$F_{\mu\nu} \to F_{\mu\nu}^{U} = U\, F_{\mu\nu}\, U^{+} \qquad (4.13)$$

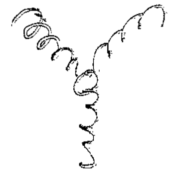
as does D_μ, see eq. (4.4), p. 66. With eq. (4.13), $\mathrm{tr}\, F_{\mu\nu} F_{\mu\nu}$ is gauge invariant.

Yang-Mills action: $\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$

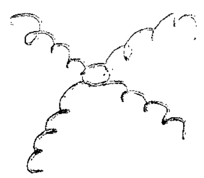
$$S_{YM}[A] = \frac{1}{2}\int d^4x\ \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}$$

$$= \frac{1}{4}\int d^dx\ F^a_{\mu\nu} F^a_{\mu\nu}$$

$$\longrightarrow \quad = \int d^dx \Big\{ \frac{1}{2}\left(\partial_\mu A^a_\nu\, \partial_\mu A^a_\nu - \partial_\mu A^a_\nu\, \partial_\nu A^a_\mu\right)$$



$$\longrightarrow \quad -g f^{abc} A^b_\mu A^c_\nu\, \partial_\mu A^a_\nu$$



$$\longrightarrow \quad + \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu \Big\}$$

(4.14)

Full action of non-Abelian gauge theory (QCD, electro-weak theory, GUT, ...) coupled to matter (leptons, quarks)

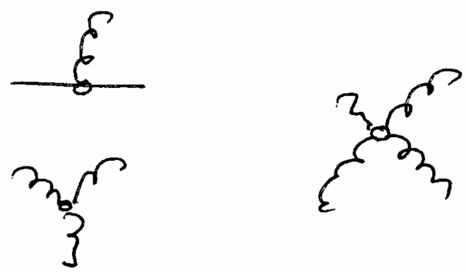
$$S[\psi, \bar{\psi}, A] = S_{YM}[A] + S_D[\psi, \bar{\psi}, A] \qquad (4.15)$$

eq. (4.14)   eq. (4.2)

with propagators and vertices:

fermions

gauge bosons



Remark: What about observables?

(i) Obervables: expectation values of <u>gauge-invariant</u> operators

$$\mathcal{O} = \langle \hat{O}[\psi, \bar{\psi}, A] \rangle \qquad (4.16)$$

with $$\boxed{\hat{O}[\psi^u, \bar{\psi}^u, A^u] = \hat{O}[\psi, \bar{\psi}, A]} \quad (4.17)$$

For example: gauge field $\langle \hat{A}_\nu \rangle$ is not an oberovable in QED ($U(1)$ gauge symmetry), but the field strength components are:

$$U(1): \quad F^u_{\mu\nu} = F_{\mu\nu}$$

$$E\text{-field}: \quad E^i = -F^{0i} = -(\partial^0 A^i - \partial^i A^0)$$

$$B\text{-field}: \quad B^i = \varepsilon^{ijk} F^{jk} \quad (4.18)$$

(ii) In a non-Abelian gauge theory (like QCD) chromo-magnetic and chromo-electric field strength components are <u>not</u> observables: $F^u_{\mu\nu} = U F_{\mu\nu} U^\dagger \neq F_{\mu\nu}$

(iii) In QCD we <u>only</u> have colour-neutral asymptotic states due to the phenomenon of confinement.

[In QED we have charge superselection sectors].

However, for high energies we have asymptotic freedom,

$$g^2/4\pi = \alpha_{YM}(E \to \infty) \to 0 \qquad (4.19)$$

and we may consider "asymptotic" gluons due to

$$gA_\nu^u = gA_\nu - \partial_\nu \omega + \underbrace{\mathcal{O}(g)}_{\to 0} \qquad (4.20)$$

More details will be given later.

Invariance of S_D in eq. (4.2):

$$S_D[\psi^U, \bar{\psi}^U, A_\nu^U] = -\int d^d x\, \bar{\psi}(x) \underbrace{U^\dagger(x)\, U(x)}_{\mathbb{1}} (\not{D} + m) U^\dagger(x) U(x) \cdot \psi(x)$$

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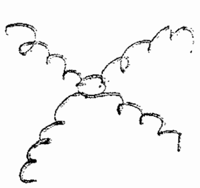

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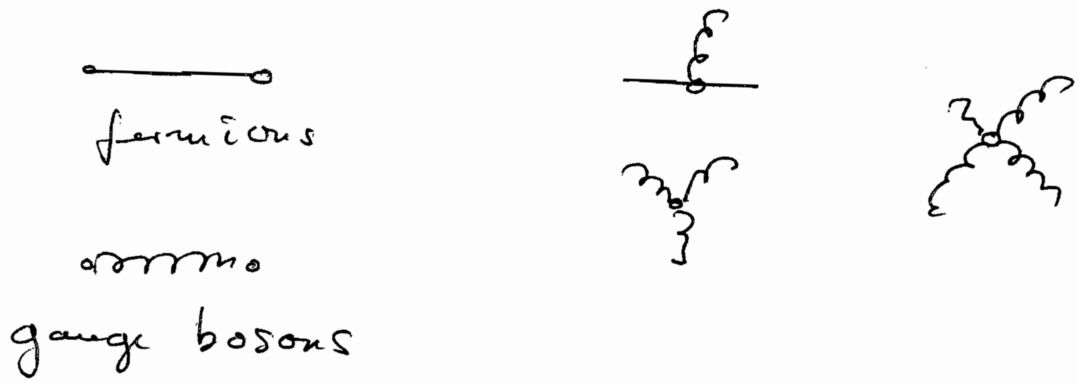

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