2 Functional integral for fermions

We want to extend the analysis of chapter 1 to fermionic fields. However, we have to take care of the fact, that fermions obey anti-commutation relations

bosonic fields	fermionic : $\bar{\Psi} = \Psi^\dagger \gamma^0$
scalars: $[\hat{\varphi}(\vec{x}), \partial_0 \hat{\varphi}(\vec{y})] = i\delta^{d-1}(\vec{x}-\vec{y})$	$\{\Psi(\vec{x}), \bar{\Psi}(\vec{y})\} = \gamma^0 \delta^{d-1}(\vec{x}-\vec{y})$
commutator	anti-commutator

see chapt. 4.2 in QFT.

2.1 Quantum mechanics

Consider the one fermion Hamiltonian:

$$H = \omega a^\dagger a \tag{2.1}$$

with the creation/annihilation operators $a^\dagger$, a respectively. The obey the anticommutation rel.

$$\boxed{\{a, a^\dagger\} = a a^\dagger + a^\dagger a = 1 \quad \& \quad a^2 = a^{\dagger 2} = 0} \tag{2.2}$$

The algebra eq. (2.2) can be represented in terms of Grassmann variables:

Grassmann algebra:

$$\boxed{c^2 = \bar{c}^2 = c\bar{c} = 0}$$ anti-commuting variables

As for the Heisenberg algebra (2.3)

$[\hat{q}, \hat{p}] = i$, we can represent eq. (2.2) by $\bar{c}$ and its derivative $\frac{\partial}{\partial \bar{c}}$, defined by

$$\boxed{\frac{\partial}{\partial \bar{c}} \bar{c} = 1} \qquad (2.4)$$

Before we go on, we collect some facts about Grassmann algebras, differentation and integration. To that end we consider n Grassmann variables c_i, $i = 1, \dots, n$.

Grassmann algebra:

$$\boxed{c_i c_j + c_j c_i = 0, \quad \forall i,j} \qquad (2.5)$$

We define

$$\boxed{\frac{\partial}{\partial c_i} c_j = \delta_{ij}.} \qquad (2.6)$$

and consistency leads to (follows from $\frac{\partial}{\partial c_i}\frac{\partial}{\partial c_j} c_n c_m$ with eq (2.5))

$$\frac{\partial}{\partial c_i}\frac{\partial}{\partial c_j} + \frac{\partial}{\partial c_j}\frac{\partial}{\partial c_i} = 0$$

$$c_i \frac{\partial}{\partial c_j} + \frac{\partial}{\partial c_j} c_i = \delta_{ij} \qquad (2.7)$$

Note also that $c_i c_j$ is not a Grassmann variable, as $(c_i c_j) c_m = c_m (c_i c_j)$ with eq. (2.5).

Integration: The functional integrals will require Grassman integration (in the completeness relations). First we note that a general fct. $f(c)$ is given by

$$\boxed{f(c) = f_0 + f_1 c} \qquad (2.8)$$

Hence, we have to define

$$\underset{\alpha}{\int dc\, 1}\ , \quad \underset{\beta}{\int dc\, c}$$

α : $$\int dc\, 1 \underset{\uparrow_{(2.6)}}{=} \int dc \frac{\partial}{\partial c} c \underset{\uparrow_{\text{absence of boundary terms}}}{=} 0 \qquad (2.9)$$

β : $$\int dc\, c = 1 \quad (\text{normalisation})$$

We conclude that Grassmann integration is equivalent to Grassmann differentation.

$$\boxed{\int dc_i\, c_j = \frac{\partial}{\partial c_i} c_j = \delta_{ij}} \qquad (2.10)$$

Change of variable: (Jacobian)

Consider $\int dc\, f(c)$

and the new variable $c = a c' + b$. With eq. (2.10) we have

$$\int dc\, f(c) = \frac{1}{a} \int dc'\, f(ac'+b) \qquad (2.11)$$

Since we have $\int dc = \int dc' \, J(c')$ with Jacobian $J(c')$, it follows that $J = \frac{1}{a}$, in contradistinction to standard integrals, where $\int dq = \int dq' \, \tilde{J}(q')$ with $\tilde{J} = a$ for $q = aq' + b$. In general we have

$$\boxed{\underbrace{dc_1 \cdots dc_n}_{\text{order important!}} = dc_1' \cdots dc_n' \; J(c') \qquad (2.12) \\ \text{with} \quad J^{-1} = \det \frac{\partial c_i}{\partial c_j'}}$$

Eq. (2.12) follows from ($c_i = a_{ij} c_j' + b_i$)

$$\int dc_1 \cdots dc_n \, f(c) = \prod_i \frac{\partial}{\partial c_i} f(c)$$

$$= \prod_{i,j} \frac{\partial c_j'}{\partial c_i} \frac{\partial}{\partial c_j'} f(c) \qquad (2.13)$$

linear nature of $c \to c'$ (anti-com. property) $\longrightarrow$

$$= \det \frac{\partial c_j'}{\partial c_i} \prod_k \frac{\partial}{\partial c_k'} f(c)$$

Gaußian Grassmann integrals: c_i complex
$\bar{c}_i = c_i^*$

Consider

$$\int dc_1 d\bar{c}_1 \cdots dc_n d\bar{c}_n \, e^{\bar{c}_i a_{ij} c_j}$$

$$= \det a \int \prod_l dc_l' d\bar{c}_l \, e^{\bar{c}_i c_i'}$$

with $c_i' = a_{ij} c_j$ and eq. (2.12). We also have

$$\int \prod_l dc_l \, d\bar{c}_l \prod_i (\bar{c}_i c_i). \qquad (2.14)$$

The integration rules eq. (2.9), (2.10) entail, that only terms of the form $\prod_i (\bar{c}_i c_i)$ contribute to the integral. All other terms either lack a specific c_i, or are prop. to $c_i^2 = 0$ after anti-commuting some Grassman variables.

Remark: The Gaußian integral

$$\mathrm{Pf}(a) := \int dc_{2n} \cdots dc_1 \; e^{\frac{1}{2} c_i a_{ij} c_j}$$

$$= \frac{1}{2^n n!} \int dc_{2n} \cdots dc_1 \left(c_i a_{ij} c_j \right)^n$$

$$= \frac{1}{2^n n!} \sum_{P} \varepsilon(P) \, a_{i_1 i_2} \cdots a_{i_{2n-1} i_{2n}} \qquad (2.15)$$

P: Permut. of $\{i_1 \dots i_{2n}\}$

with $\varepsilon(P)$ is the signature of the Permut. P.

$\mathrm{Pf}(a)$ is called Pfaffian of the matrix a. We have

$$\left(\mathrm{Pf}(a)\right)^2 = \det a \qquad (2.16)$$

exercise

Pfaffians play a crucial rôle in the path integral quantisation of Weyl and Majorana fermions.

Quantisation: Grassmann rep. of Hamiltonian H in eq. (2.1)

$$\boxed{H = \omega \, \bar{c} \frac{\partial}{\partial \bar{c}}} \qquad (2.17)$$

In analogy to bosonic QM we introduce states c with $\hat{c}|c\rangle = c|c\rangle$, normalised to delta-functions:

$$\langle c | c' \rangle = \delta(c - c') \qquad (2.18)$$

Wat is $\delta(c)$: $\int dc \, \delta(c) f(c) = f(0)$

$$\Rightarrow \boxed{\delta(c) = c} = \int d\bar{c} \, e^{\bar{c}c} \qquad (2.19)$$

Fourier repres. of δ-fct

For the path integral, i.e. $\langle c|\bar{c}\rangle$, we also need the scalar product of fcts.

$f(c) = f_0 + f_1 c$ for complex Grassmann variables c.

We define ($g = g_0 + g_1 c$)

$$(f,g) = \bar{f}_0 g_0 + \bar{f}_1 g_1 , \qquad (2.20)$$

with $\|f\|^2 = |f_0|^2 + |f_1|^2$. Eq. (2.20) leads to

$$\boxed{(f,g) = \int dc\, d\bar{c}\, e^{\bar{c}c}\, \overline{f(c)}\, g(c).} \qquad (2.21)$$

Hence we conclude $\left[|\bar{c}\rangle = \int dc'\, e^{\bar{c}c'} |c'\rangle \right]$

$$\langle c | \bar{c} \rangle = e^{\bar{c}c}. \qquad (2.22)$$

see eq. (2.19)

This leads to the fermion equivalent of the bosonic eq. for $\langle q_i | \mathcal{U}(t_i, t_{i-1}) | q_{i-1} \rangle$

$$\langle c_i | \mathcal{U}(t_i, t_{i-1} | c_{i-1} \rangle = \int d\bar{c}\, \langle c_i | \bar{c} \rangle \langle \bar{c} | \mathcal{U}(t_i, t_{i-1} | c_{i-1} \rangle$$

$$= \int d\bar{c}\, e^{\bar{c}c_i} \langle \bar{c} | \mathcal{U}(t_i, t_{i-1} | c_{i-1} \rangle$$

(2.23)

with $\mathcal{U}(t_i, t_{i-1}) \simeq e^{-\Delta t H}$, $\Delta t = t_i - t_{i-1}$

Euclidean

With $H = \omega \bar{c} \frac{\partial}{\partial \bar{c}}$ (eq. 2.17) we get for

$$\langle \bar{c} | \mathcal{U}(t, t-\Delta t) | c \rangle = \left(1 - \Delta t \, \omega \bar{c} \frac{\partial}{\partial \bar{c}} + O(\Delta t^2)\right) e^{-\bar{c}c}$$
$$= e^{-\bar{c}c(1-\Delta t\, \omega)} + O(\Delta t^2) \tag{2.24}$$

Hence, the evolution from $t_0 = -T$ to $t = T$, $T \to \infty$:

$$\langle \bar{c} | \mathcal{U}(t, t_0) | c \rangle = \lim_{n\to\infty} \int \prod_{i=1}^{n-1} dc_i \, d\bar{c}_i \; e^{-S(\bar{c}, c)} \tag{2.25}$$

with

$$S(\vec{c}, \vec{\bar{c}}) = -\sum_{i=1}^{n-1} \bar{c}_i (c_i - c_{i-1}) - \bar{c}_n c_{n-1} + \omega \frac{2T}{n} \sum_{i=1}^{n} c_i c_{i-1} \tag{2.26}$$

The continuum limit reads

$$\boxed{\langle 0 | \mathcal{U}(\infty, -\infty) | 0 \rangle = \int \mathcal{D}c \, \mathcal{D}\bar{c} \; e^{-S[c(t), \bar{c}(t)]}}$$

with action (2.27)

$$\boxed{S[c, \bar{c}] = -\int dt \left\{ \bar{c}(t) \dot{c}(t) + \omega \, \bar{c}(t) c(t) \right\}}$$

Generating Functional:

$$\boxed{Z[\eta,\bar\eta] = \int \mathcal{D}c\,\mathcal{D}\bar c\; e^{-S[c,\bar c] + \int_x (\bar\eta c - \bar c \eta)}} \tag{2.28}$$

convention

and hence

$$\boxed{\begin{aligned}&\langle T\, c(t_1)\cdots c(t_n)\,\bar c(t_{n+1})\cdots \bar c(t_{2n})\rangle \\ &= \left[\frac{1}{Z}\frac{\delta}{\delta\bar\eta(t_1)}\cdots\frac{\delta}{\delta\bar\eta(t_n)}\frac{\delta}{\delta\eta(t_{n+1})}\cdots\frac{\delta}{\delta\eta(t_{2n})}\, Z\right]_{\eta=\bar\eta=0}\end{aligned}} \tag{2.29}$$