2.2 Quantum field theory

As in QFT I we consider a

-Euclidean-Dirac action: (see QFT I, chapter

$$S_D[\psi,\bar\psi] = -\int d^dx\, \bar\psi\,(\not\partial + m)\,\psi \qquad (2.30)$$

no i

with $\boxed{\{\gamma^\mu,\gamma^\nu\} = 2\,\delta^{\mu\nu}}$ (2.31)

hermitian representation

γ_5 or better γ_{2d+1} in even dimensions 2d:

$$\boxed{\gamma_{2d+1} = \gamma_0\gamma_1\cdots\gamma_{2d-1} \Rightarrow \gamma_{2d+1}^2 = \mathbb{1}} \qquad (2.32)$$

Spin-generators: (hermitian)

$$\boxed{\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu,\gamma_\nu]} \qquad (2.33)$$

generates $SU(2)\times SU(2)$ rotations in $d=4$.

Remark: Wick rotation: $x_{0M} \to -i\, x_{0E}$

$$\Rightarrow \gamma_{0_M} \to -i\, \gamma_{0_E}$$

$$\Rightarrow \bar{\psi}_M \to -i\, \bar{\psi}_E\,, \qquad (2.34)$$

leading to eq. (2.30).

We get the functional integral in analogy

$$\boxed{Z[\eta,\bar{\eta}] = \int D\psi\, D\bar{\psi}\; e^{-S_0[\psi,\bar{\psi}] + \int_x (\bar{\eta}\psi - \bar{\psi}\eta)}} \qquad (2.35)$$

and

$$\boxed{\langle T\, \psi(x_1)\cdots\psi(x_n)\bar{\psi}(x_{n+1})\cdots\bar{\psi}(x_{2n})\rangle = \frac{1}{Z}\frac{\delta^{2n}\, Z}{\delta\eta(x_1)\cdots\delta\bar{\eta}(x_{2n})}\Bigg|_{\eta,\bar{\eta}=0}} \qquad (2.36)$$

Interacting theories with fermions are e.g.

Yukawa-type: $-\int_x \bar{\psi}(\not{\partial} + m + \underbrace{h}_{\text{Yukawa coupl}}\overbrace{\phi}^{\text{scalar field}})\psi$

QED: $-\int_x \bar{\psi}(\not{\partial} + ie\underbrace{\not{A}}_{U(1)\text{ gauge field}} + m)\psi$

QCD: $-\int_x \bar{\psi}(\not{\partial} + ig\underbrace{\not{A}}_{SU(N)\text{ gauge field}} + m)\psi$

Perturbation theory: expansion about Gaußian theory

⇒ Free fermionic generating functional:

$$Z_0[\eta,\bar\eta] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-S_0[\psi,\bar\psi]+\int_x(\bar\eta\psi-\bar\psi\eta)}$$

with (2.37)

$$S_0[\psi,\bar\psi] = -\int_x \bar\psi(\not\partial + m)\psi$$

We define

$$\psi = \psi' + \frac{1}{\not\partial+m}\cdot\eta$$
$$\bar\psi = \bar\psi' - \bar\eta\cdot\frac{1}{\not\partial+m} \qquad (2.38)$$

and get from eq. (2.37) with eq. (2.38)

$$\boxed{Z_0[\eta,\bar\eta] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-S_0[\psi,\bar\psi]} \cdot e^{\int_{x,y}\bar\eta(x)\,G_\psi(x,y)\,\eta(y)}} \qquad (2.39)$$

with

$$(\not\partial^x + m)\,G_\psi(x,y) = \delta^d(x-y) \qquad (2.40)$$

Fermionic determinant:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_0[\psi,\bar{\psi}]} = \det(\not{\partial} + m) \qquad (2.41)$$

In summary:

$$\boxed{Z_0[\eta,\bar{\eta}] = \det(\not{\partial}+m)\, e^{\int_{x,y} \bar{\eta}\, G_\psi(x,y)\, \eta(y)}} \qquad (2.42)$$

Example: 2-point fct.

$$\langle T\psi(x)\,\bar{\psi}(y)\rangle = \frac{1}{Z}\frac{\delta^2}{\delta\bar{\eta}(x)\,\delta\eta(y)} Z\Big|_{\eta,\bar{\eta}=0}$$

$$= G_\psi(x,y) \qquad (2.43)$$

Interacting theories: eg.

$$S[\psi,\bar{\psi},\varphi] = S_D[\psi,\bar{\psi},\varphi] + S_\varphi[\varphi]$$

with $\quad S_D[\psi,\bar{\psi},\varphi] = -\int_x \bar{\psi}(\not{\partial}+m+h\varphi)\psi$

(2.44)

$$\Rightarrow Z[\eta,\bar\eta,J] = \int \mathcal{D}\varphi \, \overbrace{\left[\int \mathcal{D}\psi\mathcal{D}\bar\psi \, e^{-S_D[\psi,\bar\psi,\varphi] + \int_x(\bar\eta\psi - \bar\psi\eta)}\right]}^{\text{Gaußian in fermion}} \cdot e^{-S_\phi[\varphi] + \int_x J\cdot\varphi} \qquad (2.45)$$

Gaußian fermionic integration

$$\Rightarrow Z[\eta,\bar\eta,J] = \int \mathcal{D}\varphi \, e^{-S_{eff}[\varphi] + \int_x J\varphi} \cdot e^{\int_{x,y} \bar\eta \frac{1}{\not\partial + m + h\varphi}\eta}$$

with $\qquad (2.46)$

$$S_{eff}[\varphi] = S_\phi[\varphi] - \operatorname{Tr}\ln(\not\partial + m + h\varphi)$$

In eq. (2.46) we have used the Gaußian integration as for Z_0 (take $m+h\varphi$ as the effective mass).

Note that $-\operatorname{Tr}\ln(\not\partial + m + h\varphi)$ is nothing but the fermionic part of the one-loop effective potential, see eq. (1.99), p. 34.

Note the relative minus sign, which signals the fermionic loops.

Perturbation theory: $S_\phi = \int_x \left(\frac{1}{2}\varphi(-\Delta + m_\phi^2)\varphi + V_{int}(\varphi)\right)$

$$Z[\eta,\bar\eta,J] = e^{-\int_x V\left(\frac{\delta}{\delta J}\right) + h\int_x \frac{\delta}{\delta\eta}\frac{\delta}{\delta J}\frac{\delta}{\delta\bar\eta}} \cdot e^{\frac{1}{2}\int_{x,y} J\cdot G_\phi J + \int_{x,y}\bar\eta\, G_\psi\, \eta} \qquad (2.46)$$

with

$$(-\Delta + m_\phi^2)\cdot G_\phi = \delta^d(x-y)$$
$$(\not{\partial} + \underset{= m}{m_\varphi})\, G_\varphi = \delta^d(x-y) \qquad (2.47)$$

In general

$$e^{-\int_x V\left(\frac{\delta}{\delta J}\right) + h\int_x \frac{\delta}{\delta\eta}\frac{\delta}{\delta J}\frac{\delta}{\delta\bar\eta}}$$
$$\to e^{-\int_x V\left(\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar\eta},\frac{\delta}{\delta J}\right)} \qquad (2.48)$$

Feynman rules:

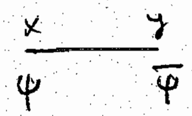
(a) Write down all diagrams in a given order N of the coupling

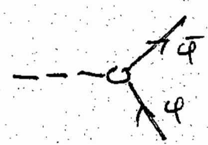
(b) Combinatorical factors

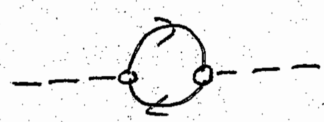
(c) (-1) for closed fermionic loops

Explanation for (c): Consider Yukawa theory

φ-prop:  : $G_\varphi(x,y)$

ψ-prop:  : $G_\psi(x,y)$

Vertex:  : h

One-loop 2-point fct.: 

$$h^2 \langle T\, \varphi(x) \int_z \bar\psi_z \varphi_z \psi_z \int_{z'} \bar\psi_{z'} \varphi_{z'} \psi_{z'}\, \varphi(y) \rangle$$

$$= -h^2 \langle T\, \varphi(x)\varphi(z) \left[\psi(z)\bar\psi(z')\right] \left[\psi(z')\bar\psi(z)\right] \varphi(z')\varphi(y) \rangle \tag{2.49}$$

↑ Grassmann prop.

Via functional integral:

$$\langle T\varphi(x)\varphi(y)\rangle_{\text{connected}} = \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} \ln Z[\eta,\bar\eta,J]\Big|_{\eta,\bar\eta,}$$

$$= \left[\frac{1}{Z}\frac{\delta^2 Z}{\delta\varphi(x)\,\delta\varphi(y)} - \left(\frac{1}{Z}\frac{\delta Z}{\delta\varphi(x)}\right)\left(\frac{1}{Z}\frac{\delta Z}{\delta\varphi(y)}\right)\right]_{\eta,\bar\eta,J=0} \tag{2.50}$$

$$\frac{\delta^2 Z}{\delta\varphi(x)\,\delta\varphi(y)} = \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}\Big[1 + \hbar\int_{x'} \frac{\delta}{\delta\eta(x')}\frac{\delta}{\delta\bar\eta(x')}\frac{\delta}{\delta J(x')}$$

$$+ \frac{\hbar^2}{2}\int_{x',x''} \frac{\delta}{\delta\eta(x')}\frac{\delta}{\delta\bar\eta(x')}\frac{\delta}{\delta J(x')}\frac{\delta}{\delta\eta(x'')}\frac{\delta}{\delta\bar\eta(x'')}\frac{\delta}{\delta J(x'')}$$

$$+ \mathcal{O}(\hbar^3)\Big]\, e^{\frac{1}{2}\int_{x,y} J\cdot G_\phi\cdot J + \int_{x,y}\bar\eta G_\psi \eta} \tag{2.51}$$

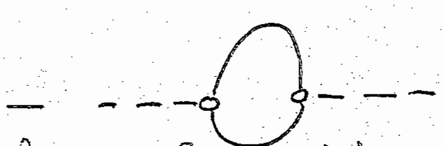
At vanishing fields the linear term (in $\hbar$) vanishes, and the $\hbar^2$-term gives the 1-loop contr.

$$\Rightarrow \boxed{\left.\frac{\delta^2 Z}{\delta\varphi(x)\,\delta\varphi(y)}\right|_{\text{1-loop}} = G_\phi(x,y) + \overset{(*)}{1}\,\hbar^2\int_{x',x''}\Big\{ G_\phi(x,x')\,G_\phi(x'',y)\Big[(-)\,G_{\psi\,\xi'\xi''}(x',x'')\cdot G_{\psi\,\xi''\xi'}(x'',x')\Big]} \tag{2.52}$$

spin-ind.

minus sign of fermionic loops

$\hbar^2$-term:



$$(*)\quad \underset{V^2}{\frac{1}{2}}\cdot\underset{(\frac{1}{2}J G_\phi J)^2}{\frac{1}{2}}\cdot\underset{(\bar\eta G_\psi \eta)^2}{\frac{1}{2}}\quad \underset{\text{ext. }\delta_J}{2}\cdot\underset{\text{int. }\delta_J}{2}\cdot\underset{V^2}{2}$$

In momentum space:

$$G_\psi(p) = \frac{1}{i\not{p} + m} = \frac{-i\not{p} + m}{p^2 + m^2}$$

← Euclidean

(2.53)

$$G_\phi(p) = \frac{1}{p^2 + m^2}$$

← Euclidean

Vacuum polarisation:

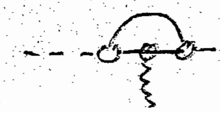
$$\xrightarrow{p} \bigcirc_{q}^{p+q} \xrightarrow{p} \simeq -h^2 \int \frac{d^dq}{(2\pi)^d} \, \mathrm{tr} \, (-i\not{q} + m)(-i(\not{p}+\not{q}) + m) \cdot \frac{1}{q^2 + m^2} \cdot \frac{1}{(p+q)^2 + m^2} \quad (2.54)$$

fermionic loop   Dirac trace

$\mathrm{tr}\,\gamma^\mu \gamma^\nu = d\,\delta^{\mu\nu}$

$$= -h^2 \int \frac{d^dq}{(2\pi)^d} \frac{\left[-d\, q\cdot(p+q) + d\, m^2\right]}{(q^2 + m^2)((p+q)^2 + m^2)}$$

Exercise

Interpretation: φ is a neutral field, but gets an 'effective' charge via virtual $\psi\bar{\psi}$-pairs (e.g. e^+e^-); in QED coupling to photon via  ~ , also