3 Functional Methods*

So far we have worked with the generating functional Z (partition function), which generates connected & disconnected correlation functions.

We have also introduced the Schwinger functional W,

$$W[J] := \ln Z[J] \qquad (3.1)$$

which generates connected Green functions (proof below).

3.1 Effective action

The effective action Γ is the generating functional of One-particle-irreducible (1PI) Green/Correlation functions (proof below).

It follows from W via a Legendre transformation: (scalar field)

$$\Gamma[\phi] := \sup_J \left[\int d^d x \, J(x) \, \phi(x) - W[J] \right] \quad (3.2)$$

We assume now that we have a maximum and differentiability (w.r.t. J, ϕ), and hence

$$\frac{\delta}{\delta J(y)} \left\{ \int_x J \cdot \phi - W[J] \right\} \Big|_{J = J_{max}} = 0$$

$$\Longrightarrow \quad \phi(y) = \frac{\delta W}{\delta J(y)} \Big|_{J_{max}} = \frac{1}{Z[J_{max}]} \frac{\delta Z}{\delta J(y)} \Big|_{J_{max}} \quad (3.3)$$

or $$\boxed{\phi(y) = \langle \varphi(y) \rangle_{J_{max}}}$$

The Legendre transform of Γ is W (strictly speaking the convex hull of W), and hence

$$\boxed{J(x) = \frac{\delta \Gamma}{\delta \phi(x)}} \quad (3.4)$$

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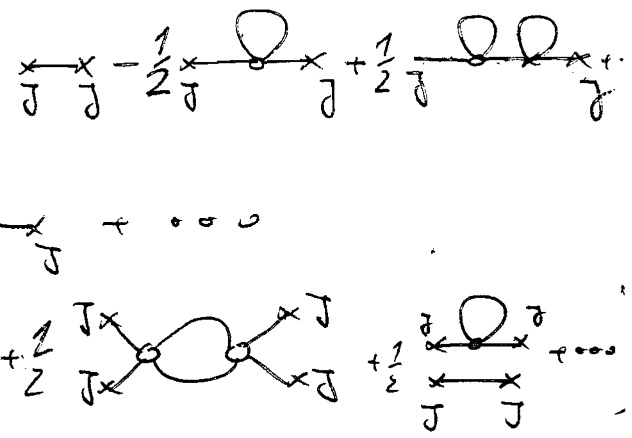
Redundancies:

$Z[J] \sim$ vac. diagr. $($ 

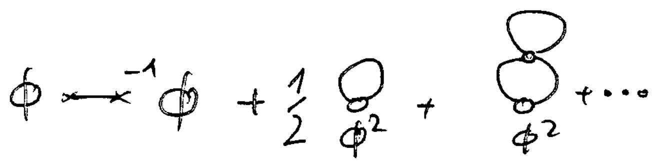
con. + discon.

$\Big\downarrow \ln$

$W[J] = \ln Z[J] \sim \ln(\text{vac. diagr.}) + \ln(1 + \cdots)$

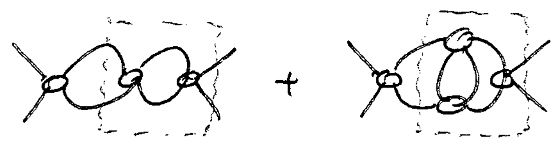
connected

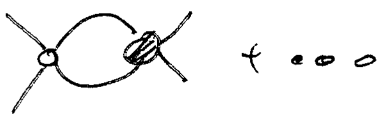
$\Big\downarrow$ Legendre

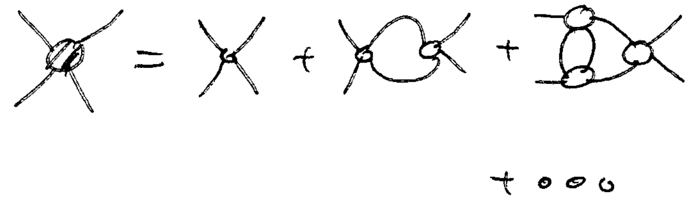
$\Gamma = \sup_J \left(\int_x J\cdot\phi - W[J] \right) \simeq$ 

1PI

$\Big\downarrow$ DSE

Closed form: eg 

$\simeq$ 

with 

What are the derivatives (moments) of Γ? We start with the observation, that

$$W^{(2)}[J]: \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = \frac{1}{Z[J]}\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)} - \frac{1}{Z[J]}\frac{\delta Z}{\delta J(x)}\frac{1}{Z[J]}\frac{\delta Z}{\delta J(y)}$$

$$= \langle \varphi(x)\varphi(y)\rangle_J - \underbrace{\langle\varphi(x)\rangle\langle\varphi(y)\rangle_J}_{\text{disconnected 2 point}}$$

$$\underset{\text{full propagator}}{W^{(2)}[J]} = \langle \varphi(x)\varphi(y)\rangle_c := \underbrace{\qquad\qquad\qquad}_{\text{connected}} \qquad (3.5)$$

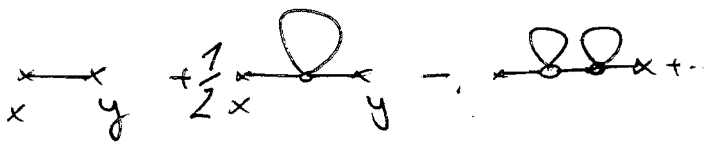
We conclude that $W^{(2)}$ is the connected 2-point function. What is $\Gamma^{(2)}[\phi] = \frac{\delta^2\Gamma[\phi]}{\delta\phi^2}$:

$$\delta^d(x-y) = \frac{\delta J(x)}{\delta J(y)} \overset{(3.4)}{=} \frac{\delta}{\delta J(y)}\frac{\delta\Gamma}{\delta\phi(x)} = \int_z \frac{\delta\phi(z)}{\delta J(y)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(x)}$$

$$(3.6)$$

$$= \int_z \frac{\delta^2 W[J]}{\delta J(y)\,\delta J(z)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(x)}$$

$$\Longrightarrow \quad \boxed{\underset{W^{(2)}(x,y)}{\langle\varphi(x)\varphi(y)\rangle_c} = \frac{1}{\Gamma^{(2)}}(x,y)} \qquad (3.7)$$

$$W^{(2)}(x,y) = \langle \varphi(x)\varphi(y)\rangle =$$ 

$$= \frac{1}{\Gamma^{(2)}}(x,y) = \frac{1}{—^{-1} - \frac{1}{2}8 + \text{2-loop}}(x,y)$$

$$\| \quad \frac{\delta^2}{\delta\phi^2}\left[-\frac{1}{2}\mathrm{Tr}\ln \rho(x)\right]_{\phi=0}$$