3.2 Functional relations

Meaning of Γ: Quantum analogue of the classical action

To see this, we derive the quantum equations of motion (Dyson-Schwinger eq.).

$$\boxed{\frac{1}{Z[J]}\int \mathcal{D}\varphi \, \frac{\delta}{\delta \phi(x)}\left[e^{-S[\varphi] + \int_{x'} J(x')\varphi(x')}\right] = 0} \quad (3.$$

(1) translat. inv. of $\mathcal{D}\varphi$
(2) no boundary term

similarily to $\int_{\mathbb{R}} dx \frac{d}{dx}\left[e^{-S(x) + J\cdot x}\right] = 0$.

Performing the derivative in eq. (3.5), we get

$$\boxed{J(x) = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J} \qquad (3.8)$$

expectation value of classical EoM.

Eq. (3.6) is the Quantum EoM, the DSE.

In terms of Γ it reads (see eq. (3.4))

$$\boxed{\frac{\delta \Gamma}{\delta \phi(x)} = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_{J_{max}(\phi)}} \qquad (3.9)$$

Computation of $\left\langle \frac{\delta S}{\delta \varphi} \right\rangle$ in ϕ^4-theory:

$$\frac{\delta S}{\delta \varphi(x)} = (-\Delta + m^2)\varphi(x) + \frac{\lambda}{3!}[\varphi(x)]^3 \qquad (3.10)$$

$$\Rightarrow \left\langle \frac{\delta S}{\delta \varphi(x)} \right\rangle \overset{\phi(x) = \langle \varphi(x) \rangle}{=} (-\Delta + m^2)\phi(x) + \frac{\lambda}{3!}\langle \varphi(x)^3 \rangle$$

$$= \frac{\delta S}{\delta \phi(x)} + \frac{\lambda}{3!}\left(\langle \varphi(x)^3 \rangle - \phi(x)^3 \right) \qquad (3.11)$$

Now we use: $\langle \varphi(x)^3 \rangle_J = \frac{1}{Z[J]} \int \mathcal{D}\varphi \; \varphi(x)^3 e^{-S[\varphi] + \int_x J \cdot \varphi}$

$$= \frac{1}{Z[J]} \frac{\delta}{\delta J(x)} \int \mathcal{D}\varphi \; \varphi(x)^2 e^{-S[\varphi] + \int_x J \cdot \varphi}$$

$\phi(x) = \frac{1}{Z[J]} \int \mathcal{D}\varphi \, \varphi(x) e^{-S + \int J \varphi}$

$= \langle \varphi(x) \rangle_J$

$$= \left(\frac{\delta}{\delta J(x)} + \phi(x) \right) \frac{1}{Z[J]} \int \mathcal{D}\varphi \; \varphi(x)^2 e^{-S[\varphi] + \int_x J \cdot \varphi}$$

$$= \left(\frac{\delta}{\delta J(x)} + \phi(x) \right) \left[W^{(2)}[J](x,x) + \phi(x)^2 \right] \qquad (3.12)$$

In general with iteration of eq. (3.12):

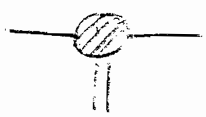
$$\boxed{\left\langle \prod_i \varphi(x_i) \right\rangle_J = \prod_i \left(\frac{\delta}{\delta J(x_i)} + \phi(x_i) \right)} \qquad (3.13)$$

Analogously to eq. (3.6), (3.7) we have

$$\frac{\delta}{\delta J(x)} = \int_y \frac{\delta \Phi(y)}{\delta J(x)} \frac{\delta}{\delta \phi(y)} \qquad (3.14)$$

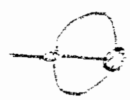
$$\text{eq. (3.7)} \longrightarrow \quad = \int_y \frac{1}{\Gamma^{(2)}[\phi]}(x,y) \frac{\delta}{\delta\phi(y)}$$

Inserting eq. (3.12) with (3.14) into eq. (3.11) we arrive at:



$$\left\langle \frac{\delta S}{\delta \varphi(x)} \right\rangle = \frac{\delta S}{\delta \phi(x)} + \frac{\lambda}{3!}\left\{ \left[\int_y \langle \varphi(x)\varphi(y)\rangle_c \frac{\delta}{\delta\phi(y)} + \phi(x)\right] \cdot \left(\frac{1}{\Gamma^{(2)}[\phi]}(x,x) + \phi^2(x)\right) - \phi^3(x) \right\}$$

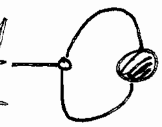
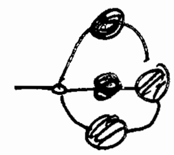
$$= \frac{\delta S}{\delta \phi(x)} - \frac{\lambda}{3!} \int_{y,z,z'} \langle \varphi(x)\varphi(y)\rangle_c \, \Gamma^{(3)}(y,z,z') \cdot \langle \varphi(x)\varphi(z)\rangle_c \langle \varphi(x)\varphi(z')\rangle_c$$



$$\frac{\delta}{\delta\phi(y)} \frac{1}{\Gamma^{(2)}}(x,x) = -\int_{z,z'} \frac{1}{\Gamma^{(2)}}(x,z)\,\Gamma^{(3)}(z,y,z')\,\frac{1}{\Gamma^{(2)}}(z',x)$$

$$+ \frac{\lambda}{2} \langle \varphi(x)\varphi(x)\rangle_c \, \phi(x) \qquad (3.15)$$

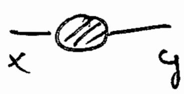
In summary:

$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = \frac{\delta S[\phi]}{\delta \phi(x)} + \frac{1}{2}$  $- \frac{1}{3!}$ 

(3.16)

with  $n>2$ $= \Gamma^{(n)}[\phi](x_1, \dots, x_n)$

 n $= S^{(n)}[\phi](x_1, \dots, x_n)$

 $= \langle \varphi(x)\, \varphi(y) \rangle = \frac{1}{\Gamma^{(2)}[\phi]}(x,y)$

or

$$\boxed{\frac{\delta \Gamma}{\delta \phi(x)} = \frac{\delta S}{\delta \varphi(x)}\left[\varphi(x) = \int_y \frac{1}{\Gamma^{(2)}[\phi]}(x,y) \frac{\delta}{\delta \phi(y)} + \phi(y)\right]}$$

(3.17)

Functional DSE

Eq (3.17) (or (3.16)) encodes the Quantum Equations of Motion (QEoM):

$$\boxed{\left.\frac{\delta \Gamma}{\delta \phi(x)}\right|_{\bar{\phi}} = 0} = \left\langle \frac{\delta S}{\delta \varphi} \right\rangle \qquad (3.18)$$

Applications:

(a) Γ generates 1PI diagrams (in ϕ)

Induction: (i) classical action is 1PI

(ii) Assume n-loop Effective action is 1PI;

Insert into (3.16) $\square$

(b) Feynman diagrams: $\frac{\delta}{\delta\phi} DSE\big|_{\phi=0}$