4.2 Generating functional

The generating functional, or here, the expectation value of observables are, in analogy to scalars and fermions, formally given by

$$\langle \hat{O}[A] \rangle = \frac{\int \mathcal{D}A \, \hat{O}[A] \, e^{-S_{YM}[A]}}{\int \mathcal{D}A \, e^{-S[A]}} \qquad (4.19)$$

pure YM

However, due to $\hat{O}[A^{\mathcal{U}}] = \hat{O}[A]$ for observables,

$\mathcal{D}A^{\mathcal{U}} = \mathcal{D}A$ and $S_{YM}[A^{\mathcal{U}}] = S_{YM}[A]$, the integral

$\mathcal{U}$ unitary

carries an infinite-dimensional redundancy.

Trivial example: $\mathcal{U}(1)$ gauge theory

with $S_{\mathcal{U}(1)}[A] = \frac{1}{4}\int d^dx \, F_{\mu\nu} F_{\mu\nu}$

$$= \frac{1}{2}\int d^dx \left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu \right) \qquad (4.20)$$

free theory

We write $A_\nu = A_\nu^{gf\,\mathcal{U}}$ with A_ν^{gf} (4.21)
satisfies some gauge. To be specific, let's take Landau gauge:

$$\partial_\nu A_\nu^{gf} = 0 \qquad (4.22)$$

and $A_\nu^{gf\,\mathcal{U}} = A_\nu^{gf} - \frac{1}{g}\partial_\nu \omega$

We perform a change of variables:

$$A_\nu \to (A_\nu^{gf}, \omega) \qquad (4.23)$$

and hence

$$\mathcal{D}A = \mathcal{D}A^{gf} \cdot \mathcal{D}\omega \cdot \mathcal{J} \qquad (4.24)$$

with Jacobi determinant $\mathcal{J}$. We implement eq. (4.24) by inserting a 1: $\mathcal{U} = e^{i\omega}$

$$1 = \Delta_{\mathcal{F}}[A] \int \mathcal{D}\omega\ \delta[\mathcal{F}[A^{\mathcal{U}(\omega)}]]$$

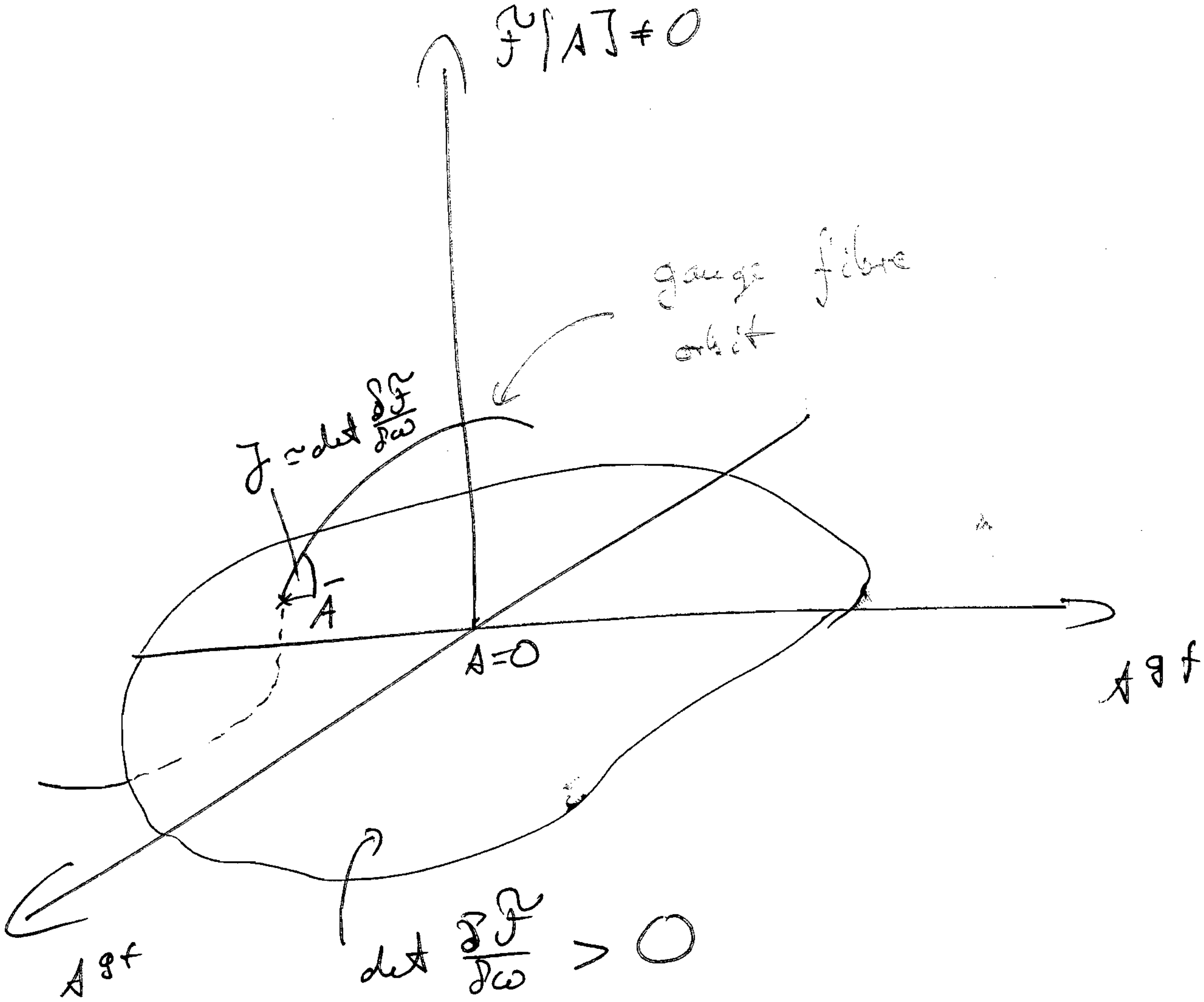
with

$$\mathcal{F} = \partial_\nu A_\nu \qquad (4.25)$$

Note that

$$\Delta_{\mathcal{F}}[A] = \left(\int \mathcal{D}\omega\ \delta[\mathcal{F}(A^{\mathcal{U}})]\right)^{-1}$$

Faddeev-Popov trick


$\tilde{F}[A] \neq 0$
gauge fibre
orbit
$J \approx \det \frac{\delta \tilde{F}}{\delta \omega}$
$\bar{A}$
$A = 0$
A^{gf}
A^{gf}
$\det \frac{\delta \tilde{F}}{\delta \omega} > 0$

$\Delta_{\mathcal{F}}[A]$ is gauge invariant by definition, as

$$\int \mathcal{D}\omega \; \delta[\mathcal{F}[(A^V)^{U(\omega)}]] \quad , \qquad V = e^{i\omega_V}$$

$$= \int \mathcal{D}\omega \; \delta[\mathcal{F}[A^{V\cdot U(\omega)}]] = \int d\omega \; \delta[\mathcal{F}[A^{U(\omega')}]]$$

$\omega' = \omega + \omega_V$

$$\underset{\mathcal{D}\omega' = \mathcal{D}\omega}{=} \int \mathcal{D}\omega \; \delta[\mathcal{F}[A^{U(\omega)}]] \qquad (4.26)$$

We use that

$$\bar{A} = \mathcal{F}[A^{U(\omega_0)}] = 0$$

$$\delta[\mathcal{F}[A^{U(\omega)}]] = \frac{1}{\left|\det \frac{\delta \mathcal{F}}{\delta \omega}\right|(A^U)} \delta[\omega - \omega_0] \qquad (4.27)$$

assuming unique solution ω_0

and hence

$$\boxed{\Delta_{\mathcal{F}}[A] = \left|\det \frac{\delta \mathcal{F}}{\delta \omega}\right|(A^{U(\omega_0)})} \qquad (4.28)$$

Faddeev-Popov det.

In Landau gauge:

$$\mathcal{F}[A^{U(\omega)}] = \partial_\nu A_\nu - \frac{1}{g}\partial_\nu \partial_\nu \omega \qquad (4.29)$$

It follows that

$$g\,\frac{\delta \mathcal{F}}{\delta \omega}(x,y) = -\partial_\nu \partial_\nu\, \delta^d(x-y) \tag{4.30}$$

and the normalisation $\Delta_{\mathcal{F}}[A]$ follows as

$$\Delta_{\partial_\nu A_\nu}[A] \simeq \left|\det\left(-\partial_\nu \partial_\nu\right)\right| \tag{4.31}$$

$$= \det\left(-\partial_\nu \partial_\nu\right)$$

$$-\partial_\nu \partial_\nu > 0$$

Finally we have for observables, eq. (4.19),

$$\langle \hat{O} \rangle = \frac{\int \mathcal{D}A\, \hat{O}[A]\, e^{-S_{U(1)}[A]}}{\int \mathcal{D}A\, e^{-S_{U(1)}[A]}} \tag{4.32}$$

$$= \frac{\int \mathcal{D}A\, \mathcal{D}\omega\, \Delta_{\mathcal{F}}[A]\, \delta[\mathcal{F}[A^u]]\, O[A]\, e^{-S_{U(1)}}}{\int \mathcal{D}A\, \mathcal{D}\omega\, \Delta_{\mathcal{F}}[A]\, \delta[\mathcal{F}[A^u]]\, e^{-S_{U(1)}}}$$

$$= \frac{\int \mathcal{D}\bar{A}\, \cancel{\mathcal{D}\omega}\, \Delta_{\mathcal{F}}[\bar{A}]\, \delta[\mathcal{F}[\bar{A}]]\, O[\bar{A}]\, e^{-S_{U(1)}}}{\int \mathcal{D}\bar{A}\, \cancel{\mathcal{D}\omega}\, \Delta_{\mathcal{F}}[\bar{A}]\, \delta[\mathcal{F}[\bar{A}]]\, e^{-S_{U(1)}}}$$

with $A = \bar{A}^{u^{-1}}$ and $\mathcal{D}\bar{A}^{u^{-1}} = \mathcal{D}\bar{A}$, $\Delta_{\mathcal{F}}[\bar{A}^{u^{-1}}] = \Delta_{\mathcal{F}}[\bar{A}]$,

$S_{U(1)}[\bar{A}^{u^{-1}}] = S_{U(1)}[\bar{A}]$ and $\mathcal{F}[\bar{A}^{u^{-1}u}] = \mathcal{F}[\bar{A}]$.

In summary

$$\boxed{\langle \hat{O} \rangle = \frac{\int \mathcal{D}A\, \delta[\mathcal{F}(A)]\, \Delta_{\mathcal{F}}[A]\, \hat{O}[A]\, e^{-S_{U(1)}[A]}}{\int \mathcal{D}A\, \delta[\mathcal{F}(A)]\, \Delta_{\mathcal{F}}[A]\, e^{-S_{U(1)}[A]}}} \quad (4.33)$$

Average gauges: instead of eq. (4.25) use

$$\text{const.} = \int \mathcal{D}\omega \mathcal{D}\mathcal{C} \left| \det \frac{\delta \tilde{\mathcal{F}}}{\delta \omega} \right|(A^u)\, \delta[\tilde{\mathcal{F}}(A^u) - \mathcal{C}]\, e^{-\frac{1}{2\xi}\int d^dx\, \mathcal{C}(x)^2}$$

$$\hookleftarrow \frac{\delta \mathcal{F}}{\delta \omega}\Big|_{\omega=0}(A^u)$$

Everything follows accordingly, only $\omega_0 = \omega_0(\mathcal{C})$

However, $\omega_0(\mathcal{C})$ drops out, and we arrive at

$$\langle \hat{O} \rangle = \frac{1}{N} \int \mathcal{D}A\, \mathcal{D}\mathcal{C} \left| \det \frac{\delta \tilde{\mathcal{F}}}{\delta \omega} \right|(A)\, \delta[\mathcal{F}(A) - \mathcal{C}]\, e^{-\frac{1}{2\xi}\int \mathcal{C}^2}$$

$$\cdot\, \hat{O}[A]\, e^{-S_{U(1)}[A]}$$

$$= \frac{1}{N} \int \mathcal{D}A\, \hat{O}[A] \left| \det \frac{\delta \tilde{\mathcal{F}}}{\delta \omega} \right|(A)\, e^{-(S_{U(1)}[A] + S_{gf}[A])},$$

with

$$S_{gf}[A] = \frac{1}{2\xi} \int d^dx\, \mathcal{F}(A)^2 \quad (4.35)$$

Remarks:

(i) $\det \frac{\delta \mathcal{F}}{\delta \omega}$ does not depend on the gauge field for $U(1)$ in linear gauges,
$\mathcal{F} = l_\nu A_\nu$ with $l_\nu = \partial_\nu, n_\nu, \ldots$ ← const. vect.
and can be dropped.

(ii) The final action

$$S[A] = S_{U(1)}[A] + \frac{1}{2\xi}\int d^dx \left(\partial_\nu A_\nu\right)^2 \qquad (4.36)$$

is that used in QFT I in the Gupta-Bleuler quantisation.

(iii) Generating functional:

$$Z[J] = \int \mathcal{D}A \left|\det \frac{\delta \mathcal{F}}{\delta \omega}\right| \cdot e^{-S_{U(1)}[A] - \frac{1}{2\xi}\int_x (\partial_\nu A_\nu)^2} \cdot e^{\int_x J_\nu A_\nu} \qquad (4.37)$$

is gauge-dependent, but

$$\langle \hat{O} \rangle = \frac{1}{Z[0]}\left(\hat{O}\left[\frac{\delta}{\delta J}\right] Z[J]\right)_{J=0}$$ are not!

Non-Abelian case: the derivation in the $U(1)$ case holds true. None of the steps was specific to $U(1)$. However, the FP-det is gauge field-dep.

up to det/g $\quad \mathcal{F}(\bar{A}) - \mathcal{C} = 0$

$$\Delta_{\mathcal{F}}[A] = \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right|_{\omega=\omega_0} \simeq |\det \mathcal{M}|[\bar{A}]$$

with

$$\mathcal{M}^{ab}(x,y) = g \frac{\delta \mathcal{F}^a(A(x))}{\delta \omega^b(y)}\bigg|_{\omega=0} = g \frac{\delta \mathcal{F}^a(A - \frac{1}{g} D\omega)}{\delta \omega^b(y)}$$

$$= -\int_z \frac{\delta \mathcal{F}^a(A(x))}{\delta A^c_\nu(z)} \frac{\delta D^{cd}_{\nu,z} \omega^d(z)}{\delta \omega^b(y)}$$

$$= -\int_z \frac{\delta \mathcal{F}^a(A(x))}{\delta A^c_\nu(z)} D^{cb}_{\nu,z} \delta^d(z-y) \tag{4.38}$$

In Lorentz gauge: $\mathcal{M}^{ab}(x,y) \simeq -\partial_\nu D^{ab}_\nu \delta^d(x-y)$

We also have, see p. 76, eq. (4.34)

$$\Delta_{\mathcal{F}}[A]\, \delta[\mathcal{F}(A) - \mathcal{C}] = |\det \mathcal{M}|[\bar{A}]\, \delta[\mathcal{F}(A) - \mathcal{C}]$$

$$= |\det \mathcal{M}|[A]\, \delta[\mathcal{F}(A) - \mathcal{C}]$$

with
$$\mathcal{M} = -\frac{\delta \mathcal{F}}{\delta A_\nu} \cdot D_\nu \tag{4.39}$$

Finally

$$\boxed{\begin{aligned} Z[J] &\simeq \int \mathcal{D}A \left|\det\left(-g\frac{\delta \mathcal{F}}{\delta A_\mu}\cdot D_\mu\right)\right| \mathbb{1}(A)\, e^{-S[A] + \int d^dx\, J^a_\mu \cdot A^a_\mu} \qquad (4.40) \\ &\text{with} \quad S[A] = S_{YM}[A] + \underbrace{\frac{1}{2\xi}\int d^dx\, \mathcal{F}^a(A)\,\mathcal{F}^a(A)}_{S_{gf}[A]} \end{aligned}}$$

Example: $\mathcal{F}^a(A) = \partial_\mu A^a_\mu$ (4.41)

$$\Rightarrow S_{gf}[A] = \frac{1}{2\xi}\int_x \left(\partial_\mu A^a_\mu\right)^2$$

FP-det:

$$\det\left(-g\frac{\delta\mathcal{F}}{\delta A_\mu} D_\mu\right) = \det\left[-\partial_\mu D_\mu\, \delta^d(x-y)\right]$$

$$= \det\left(-\partial_\mu D_\mu\right) \qquad (4.42)$$

Assume now that $\mathcal{F}(A)$ has one solution,

(does not hold for suff. smooth gauge (Gribov probl.)),

and that $D_\mu \dfrac{\delta\mathcal{F}}{\delta A_\mu}$ is positive (does not hold for ...)

Then we can use

$$\left|\det\left(-g\frac{\partial \mathcal{F}}{\partial A_\nu}\cdot \mathcal{D}_\nu\right)\right| = \det\left(-g\frac{\partial \mathcal{F}}{\partial A_\nu}\cdot \mathcal{D}_\nu\right)$$

$$= \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{-S_{gh}[c,\bar{c},A]} \qquad (4.43)$$

fermionic ghost

with

$$S_{gh}[c,\bar{c},A] = \int d^dx\, d^dy\, \bar{c}^a(x)\, g \frac{\partial \mathcal{F}^a}{\partial A_\nu^c}\cdot \mathcal{D}_\nu^{cb}\, c^b(y)$$

For Lorentz gauge:

$$S_{gh}[c,\bar{c},A] = \int d^dx\; \bar{c}^a \partial_\nu \mathcal{D}_\nu^{ab} c^b \qquad (4.44)$$

Comments:

(i) the ghost has '<u>negative</u>' propagator

$$G_c(p) = -\frac{1}{p^2}\delta^{ab} \qquad (4.45)$$

(ii) the ghost does not obey the spin-statistics theorem

In summary,

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar c\; e^{-S[A,c,\bar c]} \cdot e^{\int_x \left(J^a_\mu A^a_\mu + \bar\eta c - \bar c \eta\right)} \tag{4.46}$$

with ($\mathcal{F}^a(A) = \partial_\mu A^a_\mu$)

$$S[A,c,\bar c] = S_{YM}[A] + S_{gh}[c,\bar c,A] + S_{gf}[A]$$

$$S_{YM}[A] = \frac{1}{4}\int_x F^a_{\mu\nu} F^a_{\mu\nu}$$
$$S_{gh}[c,\bar c,A] = \int_x \bar c^a\, \partial_\mu D^{ab}_\mu c^b \tag{4.47}$$
$$S_{gf}[A] = \frac{1}{2\xi}\int_x \left(\partial_\mu A^a_\mu\right)^2$$

and $F^a_{\mu\nu}$ given in eq. (4.11),

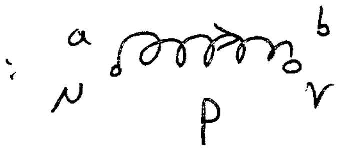
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^c_\mu A^c_\nu \tag{4.48}$$

For a general gauge $\mathcal{F}^a(A)$ we have

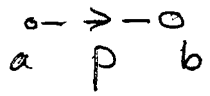
$$S_{gh} = \int \bar c\, \partial_\mu \frac{\delta \mathcal{F}}{\delta A_\mu}\, c\,, \quad S_{gf} = \frac{1}{2\xi}\int \mathcal{F}^{a\,2} \tag{4.49}$$

Feynman rules: (in Lorentz gauge)

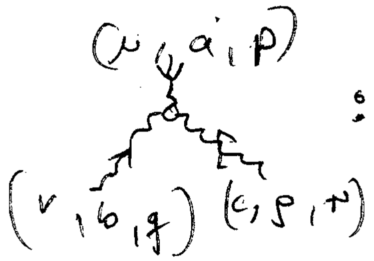
Propagators:

gluon: 

: $\frac{1}{p^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}(1-\xi)\right)\delta^{ab}$ (4.50a)

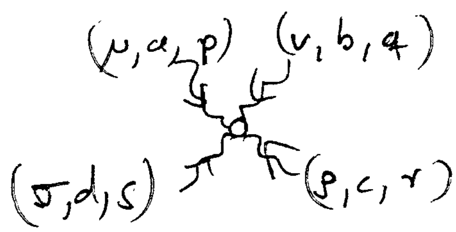
ghost: 

: $-\frac{1}{p^2}\delta^{ab}$ (4.50b)

Vertices: 

: $ig f^{abc}$ (4.51a)

$\cdot\left(\delta_{\mu\nu}(p-q)_\rho + \delta_{\nu\rho}(q-r)_\mu + \delta_{\rho\mu}(r-p)_\nu\right)$

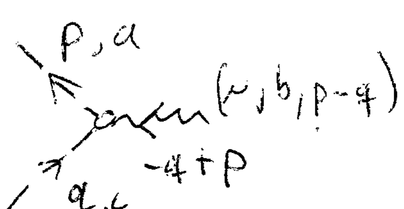


: $-g^2 f^{abe} f^{cde} (2\pi)^d \delta^d(p+q+r+s)$

$\cdot\left(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}\right.$

$\left. + \text{cycl. permuts in } \nu\rho\sigma\right)$

(4.51b)



: $-g f^{abc} p_\mu$ (4.52)

Gauge invariance: the generating functional in eq. (4.46) is built upon the gauge-fixed action eq. (4.47). How does gauge-invariance manifest itself?

(i) Consider observable $\langle \hat{O}[A] \rangle$ with $\hat{O}[A^u] = \hat{O}[A]$: (4.53)

$$\frac{1}{N}\int \mathcal{D}A\, \hat{O}[A]\, e^{-S_{YM}[A]} = \frac{1}{N}\int \mathcal{D}A\, \hat{O}[A^u]\, e^{-S_{YM}[A]}$$

More generally: $f(A^u) \neq f(A)$ ($\langle f(A)\rangle$ no observ.)

$$\frac{1}{N}\int \mathcal{D}A\, f(A^u)\, e^{-S_{YM}[A]} \overset{A=\tilde{A}^{u^{-1}}}{=} \frac{1}{N}\int \mathcal{D}\tilde{A}^{u^{-1}}\, f(\tilde{A}^{u^{-1}u})\, e^{-S_{YM}[\tilde{A}^{u^{-1}}]}$$

$$= \frac{1}{N}\int \mathcal{D}\tilde{A}\, f(\tilde{A})\, e^{-S_{YM}[\tilde{A}]} \qquad (4.54)$$

$$\Rightarrow \boxed{\frac{1}{N}\int \mathcal{D}A\, [f(A^u) - f(A)]\, e^{-S_{YM}[A]} = 0} \qquad (4.55)$$

gauge inv. of S_{YM} (and of $\mathcal{D}A$)