4.4 Quantum master equation*

The generating functional shows now BRST invariance,

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}C\,\mathcal{D}\bar{C}\,\mathcal{D}b\, e^{-S[A,C,\bar C,b]} \cdot e^{\int_x (J\cdot A + \bar\eta\cdot C - \bar C \eta)}$$

$$= \int \mathcal{D}A\,\mathcal{D}C\,\mathcal{D}\bar{C}\,\mathcal{D}b\, e^{-S[A,C,\bar C,b]} \cdot e^{\int_x \{J\cdot(A+\varepsilon QA) + \bar\eta(C+\varepsilon QC) - (\bar C+\varepsilon Q\bar C)\eta\}} \tag{4.92}$$

$$\Rightarrow \boxed{\int_x \{ J\cdot\langle \varepsilon Q A\rangle + \bar\eta\langle\varepsilon Q C\rangle - \langle\varepsilon Q\bar C\rangle\eta\} = 0} \tag{4.93}$$

Again this is seemingly a complicated loop equation, as the expectation values are non-trivial. However, the derivative property of Q, $Q^2=0$ in eq. (4.85) comes to our aid:

We add source terms for $QA, QC, Q\bar C$ in Z!

They do not transform under Q due to $Q^2=0$.

We then have the generating functional

$$Z[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}] \tag{4.94}$$
$$= \int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\mathcal{D}b\, e^{-S + \int_x \{ J\cdot A + \bar{\eta} c - \bar{c}\eta + L_A \cdot QA + L_c Qc + L_{\bar{c}} Q\bar{c} \}}$$

$Q\bar{c} = +b$: Source term for b

and eq. (4.93) still holds. The expectation values can be represented as

$$\langle QA \rangle = \frac{1}{Z}\frac{\delta Z}{\delta L_A}, \quad \langle Qc \rangle = \frac{1}{Z}\frac{\delta Z}{\delta L_c}$$
$$\langle Q\bar{c} \rangle = \frac{1}{Z}\frac{\delta Z}{\delta L_{\bar{c}}} \left(= \langle b \rangle \right) \tag{4.95}$$

and we get for eq. (4.93):

$$\boxed{\int_x \left\{ J \circ \frac{\delta Z}{\delta L_A} - \bar{\eta} \circ \frac{\delta Z}{\delta L_c} - \frac{\delta Z}{\delta L_{\bar{c}}} \circ \eta \right\} = 0} \tag{4.96}$$

In (4.96) we have used that $\bar{\eta}\varepsilon = -\varepsilon\bar{\eta}$, see eq. (4.93). Note also that $\frac{\delta Z}{\delta L_{\bar{c}}} = \langle b \rangle$. For $L_{\bar{c}} = 0$ this is simply $\langle b \rangle \underset{\substack{\uparrow \\ \text{Lorentz gauge}}}{=} \partial_\mu \bar{A}$ with $\bar{A} = \langle A \rangle$.

Finally we define the effective action, that generates 1PI Green fcts.: (see chapter 3.1, p.5

$$\boxed{\Gamma[A, c, \bar{c}; L_A, L_c, L_{\bar{c}}] = \Big\{\int d^dx \; J\cdot A + \bar{\eta}\cdot c - \bar{c}\cdot\eta - \ln Z[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}]\Big\}} \qquad (4.97)$$

with

$$J = \frac{\delta\Gamma}{\delta A}, \quad \bar{\eta} = -\frac{\delta\Gamma}{\delta c}, \quad \eta = -\frac{\delta\Gamma}{\delta\bar{c}} \qquad (4.98)$$

This leads to the master equation

Grassmann

$$\boxed{\int d^dx \left\{\frac{\delta\Gamma}{\delta L_A}\cdot\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L_c}\cdot\frac{\delta\Gamma}{\delta c} + \frac{\delta\Gamma}{\delta L_{\bar{c}}}\cdot\frac{\delta\Gamma}{\delta\bar{c}}\right\} = 0} \qquad (4.99)$$

non-Grassmann

which encodes BRST-invariance of the effective action.

Remarks & examples:

(i) Classically, the effective action is the classical action, eq. (4.72),

$$S[A, C, \bar{C}, b; L_A, L_C, L_{\bar{C}}] = S_{YM}[A] + S_{gh}[C, \bar{C}, A] + S_{gf}[A, b] - \int_x \{ L_A \cdot QA + L_C \cdot QC + L_{\bar{C}} Q\bar{C} \} \tag{4.100}$$

It follows that

$$\frac{\delta S}{\delta L_A} = -QA, \quad \frac{\delta S}{\delta L_C} = -QC, \quad \frac{\delta S}{\delta L_{\bar{C}}} = -Q\bar{C} \tag{4.101}$$

and hence: $(\Sigma) \int_x \left\{ QA \cdot \frac{\delta S}{\delta A} + QC \frac{\delta S}{\delta C} + Q\bar{C} \frac{\delta S}{\delta \bar{C}} \right\} = 0$ (4.102)

which can be rewritten as $\boxed{\delta_\Sigma S = 0}$.

Hence, eq. (4.99) implies classical BRST invariance.

(ii) The generator of quantum BRST-transformations is

$$(S_\Gamma :=) \int_x \left\{ \frac{\delta \Gamma}{\delta L_A} \cdot \frac{\delta}{\delta A} + \frac{\delta \Gamma}{\delta L_C} \cdot \frac{\delta}{\delta C} + \frac{\delta \Gamma}{\delta L_{\bar{C}}} \frac{\delta}{\delta \bar{C}} \right\} \tag{4.103}$$

(iii) Quantum action principle:

(1) classical: $\delta_\varepsilon S = 0$

(2) 1-Loop: $$\frac{\delta S}{\delta L_A}\circ\frac{\delta\Gamma}{\delta A}\bigg|_{1\text{-Loop}} + \frac{\delta\Gamma}{\delta L_A}\bigg|_{1\text{-Loop}}\cdot\frac{\delta S}{\delta A} + \cdots = 0 \qquad (4.104)$$

$\vdots$

n-Loop: $$\frac{\delta\Gamma}{\delta L_A}\bigg|_{n-1\text{-Loop}}\circ\frac{\delta\Gamma}{\delta A}\bigg|_{n\text{-Loop}} + \frac{\delta\Gamma}{\delta L_A}\bigg|_{n\text{-Loop}}\cdot\frac{\delta\Gamma}{\delta A}\bigg|_{n-1\text{-Loop}} + \cdots = 0$$

(iv) Integrating out b:

$$\frac{\delta\Gamma}{\delta L_{\bar c}}\bigg|_{L_{\bar c}=0} = -\langle b\rangle_{L_{\bar c}=0} = -\frac{1}{\xi}\partial_\mu A_\mu$$

$$\Rightarrow \boxed{\int_x \left\{ \frac{\delta\Gamma}{\delta L_A}\circ\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L_c}\frac{\delta\Gamma}{\delta C} - \frac{1}{\xi}\partial_\mu A_\mu\cdot\frac{\delta\Gamma}{\delta\bar C} = 0\right.} \quad (4.105)$$

The anti-ghost only appears linearly in the generating functional Z. Due to translation invariance (DSE) it follows that $\left(\int DA\, DC\, D\bar C\, Db \frac{\delta}{\delta\bar C}\cdot = 0\right)$

$$\langle \partial_\mu D_\mu C\rangle + \eta = 0$$

or in terms of Γ: $$\boxed{\partial_\mu\frac{\delta\Gamma}{\delta L_{A_\mu}} - \frac{\delta\Gamma}{\delta\bar C} = 0} \qquad (4.105b)$$

Putting eq.(4.105a) and eq.(4.105b) together we arrive at

$$\boxed{\int_x \left\{ \frac{\delta \Gamma}{\delta L_A} \cdot \frac{\delta \Gamma}{\delta A} + \frac{\delta \Gamma}{\delta L_c} \circ \frac{\delta \Gamma}{\delta C} - \frac{1}{\xi} \partial_\mu A_\mu \, \partial_\nu \frac{\delta \Gamma}{\delta L_{A\nu}} \right\} = 0} \qquad (4.105c)$$

In eq. (4.105c) the b-field is integrated out evidently the variation of the anti-ghost simply amounts to a gauge transformation of the gauge fixing term (as it was introduced in eq. (4.77) in the first place.

(v) Transversality: consider the (amputated) 2-point correlation function $\frac{\delta^2\Gamma}{\delta A^2}$.

We take a $\frac{\delta}{\delta A^a_\mu(x)}\frac{\delta}{\delta c^b(y)}$-derivative of (4.99) (or (4.105a) at vanishing fields and BRST-sources:

$$\boxed{\int_z \left\{ \frac{\delta^2\Gamma}{\delta c^b(y)\,\delta L^d_{A_\nu}(z)} \cdot \frac{\delta^2\Gamma}{\delta A^d_\nu(z)\,\delta A^a_\mu(x)} + \frac{\delta^2\Gamma}{\delta L^d_{\bar c}(z)\,\delta A^a_\mu(x)} \frac{\delta^2\Gamma}{\delta c^b(y)\,\delta \bar c^d(z)} \right\} = 0 \qquad (4.106)}$$

Integrating out b, see p. 102, 102a

with $\left(\frac{\delta^2\Gamma}{\delta L_{\bar c}\,\delta A_\mu}\right)^{da}_\mu(z,x) \overset{\downarrow}{=} -\frac{1}{\xi}\,\frac{\delta\, \partial_\nu A^d_\nu(z)}{\delta A^a_\mu(x)}$.

$$\left(\frac{\delta^2\Gamma}{\delta c\,\delta\bar c}\right)^{bd}(y,z) \underset{\text{eq. (4.105b)}}{=} \frac{\delta}{\delta c^b(y)}\,\partial_\nu \frac{\delta\Gamma}{\delta L^d_{A_\nu}(z)} = \partial^z_\nu \frac{\delta^2\Gamma}{\delta c^b(y)\,\delta L^d_{A_\nu}} \qquad (4.107)$$

In summary:

$$\boxed{\int_z \frac{\delta^2\Gamma}{\delta L^d_{A_\nu}(z)\,\delta c^b(y)}\left\{\frac{\delta^2\Gamma}{\delta A^d_\nu(z)\,\delta A^a_\mu(x)} + \overbrace{\frac{1}{\xi}\,\partial_\mu\partial_\nu\,\delta^d(z-x)}^{-\frac{\delta^2 S_{gf}[A]}{\delta A^d_\nu(z)\,\delta A^a_\mu(x)}}\right\} = 0}$$

with $\left(\frac{\delta^2\Gamma}{\delta L_A\,\delta c}\right)^{db}_\nu(z,y) = \partial_\nu\,\delta^d(z-y)\,\delta^{db} + (1\text{-loop}) + \dots \qquad (4.108)$