6.4 The continuum limit of lattice Yang-Mills

In the continuum limit of a lattice theory we keep the physical mass a (smallest phys. momentum scale) fixed. This implies that

$$\hat{m} = m \cdot a \to 0 \qquad (6.58)$$

and hence the correlation length $\hat{\xi} = 1/\hat{m}$ diverges. This is the signature of a 2nd order phase transition. At this point the system has infinite many points inside a physical distance.

In lattice YM we only have g_0 for tuning this limit:

$$\boxed{\hat{\xi}(g_0) \xrightarrow[g_0 \to g_*]{} \infty \qquad (6.59)}$$

This entails for a general observable $\mathcal{O}$,

$$\mathcal{O}(g_0, a) = \left(\frac{1}{a}\right)^{d_{\mathcal{O}}} \cdot \hat{\mathcal{O}}(g_0) \qquad (6.60)$$

where $d_{\mathcal{O}}$ is the momentum dimension of $\mathcal{O}$.
In the continuum limit we then have

$$\mathcal{O}(g_0 \to g_*, a \to 0) = \mathcal{O}_{phys} \qquad (6.61)$$

We conclude that if we know the functional dep. of $\mathcal{O}$ on g_0, we know $g_0(a)$ with $\mathcal{O}(g_0(a), a) = \mathcal{O}_{phys}$.

Remark: The above argument seems to imply that $g_0(a)$ depends on the choice of $\mathcal{O}$. However, it turns out, that for sufficiently small a, $g_0(a)$ is universal (up to sub-leading terms ($\leftarrow$ renorm. group eqs).

Let us now take as $\mathcal{O}$ the $q\bar{q}$-pot. V defined in the previous section in eq. (6.55).

$$V(L, g_0, a) = \frac{1}{a}\hat{V}(\hat{L}, g_0) \qquad (6.61)$$

Now, keeping $V(L, g_0, a)$ fixed at its physics value V_{phys} while $a \to 0$ implies

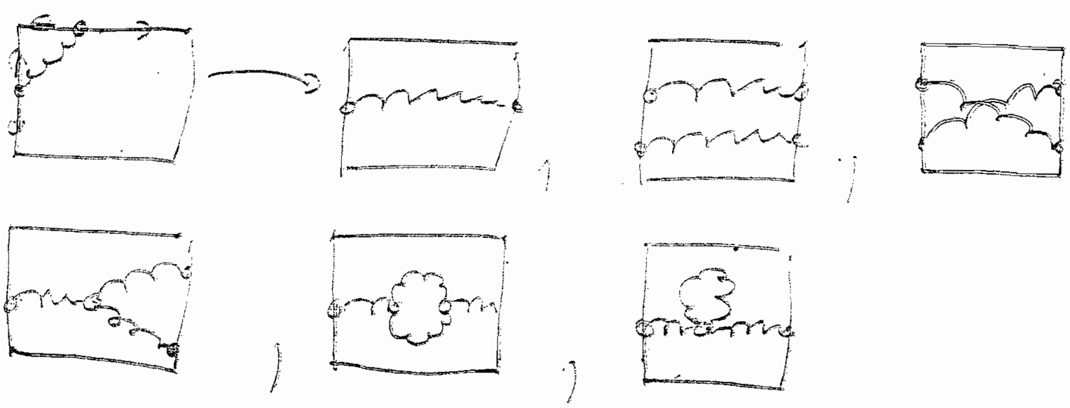
$$\boxed{\left(a\frac{\partial}{\partial a} - \beta(g_0)\frac{\partial}{\partial g_0}\right) V(L; g_0, a) = 0} \qquad (6.62)$$

$\uparrow \Lambda \sim 1/a$
cut-off scale (coarse graining scale)

with $\boxed{\beta(g_0) = -a\frac{\partial g_0}{\partial a}}$ $\qquad (6.63)$

$(\Lambda \sim 1/a : \Lambda\partial_\Lambda = -a\partial_a)$

Up to g_0^4, we have the diagrams



The pot. is computed to

$$V(L) = -\frac{g_0^2(a)}{4\pi L}\, C_2(\text{fund})\left[1 + g_0^2(a)\,\frac{11N}{24\pi^2}\ln\hat{L} + \text{const.}\right] \quad (6.64)$$

$(N^2-1)/2N$ (p. 125)

see eq. (5.57), p. 128. If we insert eq. (6.64) into eq. (6.63) we get

$$\beta(g_0) = -\frac{g_0^3}{16\pi^2}\,\frac{11}{3}N = \beta_0\, g_0^3 \quad (6.65)$$

as in eq. (5.57). Since $\beta(g_0)$ is smaller 0 (asympt. freedom), the coupling is driven to zero in the limit $a \to 0$:

$$\boxed{a = 1/\Lambda_L \; e^{\frac{1}{2\beta_0 g_0^2}}} \quad (6.66)$$

(physical) Reference scale

Let us now rewrite the RG-equation eq. (6.62) in terms of physical scales:

$$\left(L\frac{\partial}{\partial L} + \beta(g_0)\frac{\partial}{\partial g_0}\right) V(L, g_0, a) = -V(L, g_0, a)$$

(6.67)

or $$\left(L \frac{\partial}{\partial L} + \beta(g_0) \frac{\partial}{\partial g_0} \right) \underbrace{\tilde{V}(L, g_0, a)}_{= L \cdot V(L, g_0, a)} = 0 \qquad (6.68)$$

Eq. (6.68) entails that a change in the physical distance L can be absorbed in a corresponding change of the bare coupling g_0. We can write

$$\tilde{V} = \tilde{V}(L, \bar{g}(L), a) \qquad (6.69)$$

with $\bar{g}(L)$ has the property $L \frac{\partial}{\partial L} \bar{g} = -\beta(\bar{g}(L))$, and hence

$$\bar{g}^2(L) = \frac{\bar{g}_0^{\,2}}{1 + \beta_0 \bar{g}_0^2 \ln L^2/a^2} \qquad (6.70)$$

see eq. (5.58). Using eq. (6.70) in $\tilde{V}$ leads to

$$V(R) = - C \cdot \frac{\alpha_S(L)}{L} \qquad (6.71)$$

with

$$\alpha_S(L) = \frac{\bar{g}^2(L)}{4\pi} \qquad (6.72)$$

This seemingly depends on a, but by using eq. (6.66) we arrive at

$$\boxed{\begin{aligned} \alpha_s(L) &= \frac{1}{4\pi} \frac{\bar{g}_0^2}{1 + \beta_0 \bar{g}_0^2 \ln L^2 \Lambda_L^2 e^{-1/(\beta_0 \bar{g}_0^2)}} \\ &= \frac{1}{4\pi} \frac{1}{\beta_0 \ln L^2 \Lambda_L^2} \end{aligned}} \qquad (6.73)$$

Remarks:

(1) $\bar{g}_0^2/4\pi$ is the running coupling α_s at the scale $L = a$: $\alpha_s(a) = \frac{\bar{g}_0^2}{4\pi}$

(2) The scale Λ_L is the momentum scale at which the (one-loop) coupling diverges: $\alpha_s(L \to 1/\Lambda_L) \to \infty$.

$\Lambda_{QCD_{Latt}} = \frac{1}{83.5} \Lambda_{QCD_{MOM}}$

(3) Λ_L is RG-invariant

$$\boxed{a \frac{d}{da} \Lambda_L = \left(a \frac{\partial}{\partial a} - \beta(g_0) \frac{\partial}{\partial g_0} \right) \left(\frac{1}{a} e^{\frac{1}{2\beta_0 \bar{g}_0^2}} \right) = 0} \qquad (6.74)$$

asymptotic scaling*: Finally we would like to know close we are already to the continuum with an Observable $\hat{O}$, eq.(6.60)

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higher accuracy: Consider the 2-loop β-function:

$$\beta = -\beta_0 g_0^3 - \beta_1 g_0^5 + O(g_0^7) \qquad (6.75a)$$

with

$$\beta_0 = \frac{11}{16\pi^2} \quad , \quad \beta_1 = \frac{1}{(16\pi^2)^2} 102 \qquad (6.75b)$$

leading to (analogue to eq. 6.66),

$$a = 1/\Lambda_L \cdot \hat{L}(g_0)$$

with $$\hat{L}(g_0) = \left(\beta_0 g_0^2\right)^{-\beta_1/2\beta_0^2} e^{-\frac{1}{2\beta_0 g_0^2}}$$

$$(6.76)$$

We conclude from eq.(6.61) and $a \frac{d}{da} \mathcal{O}\Big|_{a \to 0} = 0$ that

$$\boxed{\hat{O} \simeq \hat{C}_{\mathcal{O}} \left[\hat{L}(g_0)\right]^{d_{\mathcal{O}}}} \qquad (6.77)$$

The behaviour in eq. (6.77) signals the continuum limit and is called 'asymptotic scaling':

(1) If g_0 and hence a becomes too small (at fixed lattice size), the physics scales will eventually exceed the lattice size (finite size effects)

(2) If g_0 gets to big, the lattice will get too coarse, and eq. (6.77) is not valid any more.

In summary continuum physics is only seen in the (narrow) window avoiding (1) & (2).

string tension: In the last chapter we have derived (for $g_0 \to \infty$) a linearly rising potential (confinement) with

$$\hat{V}(\hat{L}) = \hat{\sigma}\,\hat{L} \quad \text{with } \hat{\sigma} = -\ln \overset{\overset{1/g_0^2}{\shortparallel}}{\beta}/18$$

see eq. (6.55), (6.56). In the continuum limit such a potential has the physical string tension

$$\sigma = \lim_{a\to 0} \frac{1}{a^2}\,\hat{\sigma}(g_0(a)) \qquad (6.78)$$

and hence

$$\boxed{\hat{\sigma} \simeq \hat{C}_\sigma \cdot \left[\hat{L}(g_0)\right]^2 \qquad (6.79)}$$

Note that $\hat{L}(g_0)^2$ is non-perturbative and vanishes in any order of pert. theory.