5.2 Running coupling

for QED see QFT I, p. 195-20

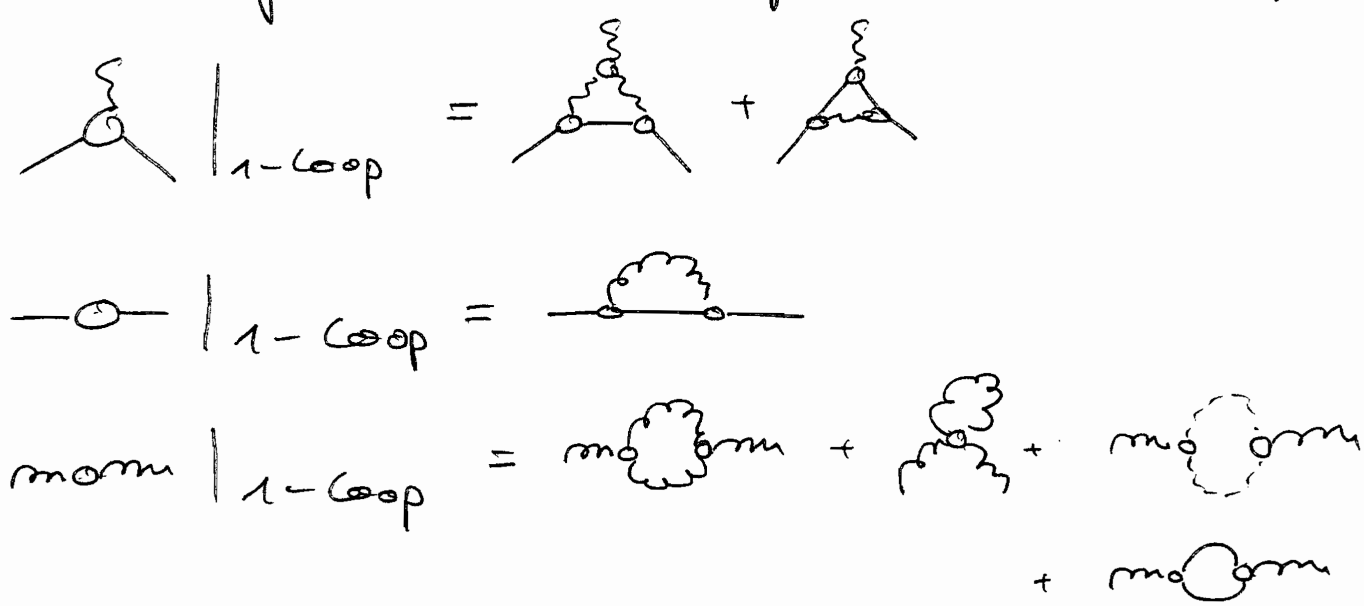
The renormalised coupling

$$g = g_0 / Z_g \tag{5.21}$$

is computed e.g. from the ratio

$$Z_g = Z_{1,F} \Big/ Z_2 \, Z_3^{1/2} \tag{5.22}$$

that is from the diagrams (at 1-loop)



In dim. reg. this leads to diagrams

with $$\mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} \left[\text{loop integrand} \right]$$

that is, $Z = Z(\mu)$.

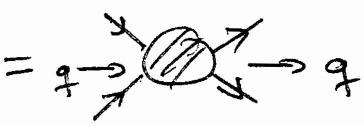
The bare parameters do <u>not</u> depend on μ, and hence

$$\boxed{\beta_g = \mu \frac{d}{d\mu} g = - g \, \mu \frac{d}{d\mu} \ln Z_g} \qquad (5.21)$$

However, Observables $\mathcal{O}$ do also <u>not</u> depend on μ:

$$\boxed{\mu \frac{d}{d\mu} \mathcal{O} = 0} \qquad (5.22)$$

Apply Eq. (5.22) to a ~~cross~~-section, e.g.

$$\mathcal{O}(g^2, q^2/\mu^2) =$$ 

No dep. on quark mass: massless quarks
or heavy quark limit: $m \to \infty$
(quenched)

and with eq. (5.22) we have

$$\mu \frac{d}{d\mu} \mathcal{O} = \boxed{\left(\mu \frac{\partial}{\partial\mu} + \beta \partial_g\right) \mathcal{O} = 0} \qquad (5.23)$$

RG-equation

Now, if we choose $\mu^2 = q^2$:

$$\mathcal{O}(g^2, q^2/\mu^2) = \mathcal{O}(4\pi\alpha_s, 1) \qquad (5.24)$$

with running coupling

$$\boxed{\alpha_s(q^2) = \frac{g^2(q^2)}{4\pi}} \qquad (5.25)$$

Remark:

(1) In QFT I we have seen that we could pick the renormalisation condition $g^2\big|_{q^2=\mu^2} = g^2_{\text{phys}}$

By choosing $\mu^2 = q^2$ this is then valid at all scales.

(2) The sign of the β-function in eq. (5.21) then tells us, whether the interaction strength grows ($\beta > 0$) or gets smaller ($\beta < 0$) for larger momenta.

Rehash dimensional regularisation: e.g.

QFT I, p.187

$$I_{\mu_1\mu_2} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^{2n}} \frac{1}{(l+p)^{2m}} T_{\mu_1\mu_2}(p,l)$$

with $I_{\mu_1\mu_2}(p,l) = \{ \delta_{\mu_1\mu_2}, p_{\mu_1}p_{\mu_2}, l_{\mu_1}l_{\mu_2}, p_{\mu_1}l_{\mu_2} \}$

$$I_{\mu_1\mu_2} \to \mu^{2\varepsilon} \int \frac{d^d l}{(4\pi^2)^{d/2}} \frac{1}{l^{2n}} \frac{1}{(l+p)^{2m}} T_{\mu_1\mu_2}(p,l)$$

with $d = 4-2\varepsilon$

Remarks:

(i) $[I_{\mu_1\mu_2}]_{d=4} = [I_{\mu_1\mu_2}]_{d=4-2\varepsilon}$

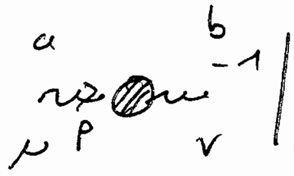
since $[\mu^{2\varepsilon} d^d l] = 4 \quad \forall d$

(ii) dependence on new scale μ.

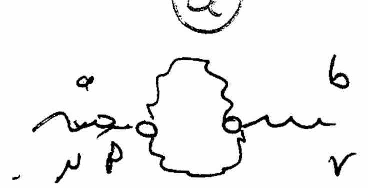
For the running coupling we need e.g. $Z_{1,F}$ (quark self-energy), Z_2 (quark-gluon vertex) and Z_3 (gluon vac. pol.).

<u>One loop computation:</u> (explicit in Feynman (Feynman rules, p. 82, 107) gauge)

Vacuum polarisation:



$^{-1}\Big|_{1\text{-loop}} \simeq$

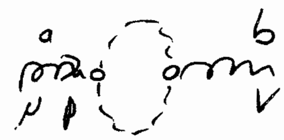


(a)

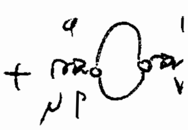
$\| $

$\left(p^2\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$ $+$

$+$

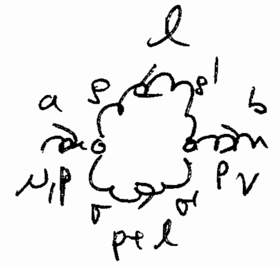


$+$

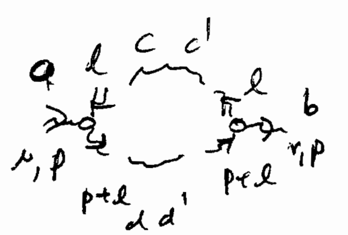


(b) (c) (d)

(a)



$=$



$\leftarrow$ Feynman rules p. 82

$$(a) = \frac{1}{2}\underbrace{g^2}_{g \text{ in } d \text{ dim}}\mu^{2\varepsilon}\int\frac{d^d l}{(2\pi)^d} f^{acd}\left[\delta_{\mu\rho}(p-l)_\sigma + \delta_{\rho\sigma}(2l+p)_\mu + \delta_{\sigma\mu}(-2p-l)_\rho\right]\frac{\delta_{\rho\rho'}}{l^2} f^{bcd}\left[\delta_{\nu\rho'}(-p+l)_{\sigma'} + \delta_{\rho'\sigma'}(-2l-p)_\nu + \delta_{\sigma'\nu}(2p+l)_{\rho'}\right]\frac{\delta_{\sigma\sigma'}}{(l+p)^2}\Big\} \quad (5.26)$$

Traces & contractions:

$f^{bdc} = -f^{bcd}$

$$f^{acd} f^{bcd} = \left(-i f^{a}\right)^{cd} \left(-i f^{b}\right)^{dc} \tag{5.27}$$

$$= \mathrm{tr}\, t^a t^b \qquad \text{with } \left(t^a\right)^{bc} = -i f^{abc}$$

t^a are generators in the adjoint ($N^2-1 \times N^2-1$ dim) representation of the gauge group.

[The t^a satisfy the Lie-algebra (see exercise sheet) as $f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$.

In general we have

Dynkin index

$$\mathrm{Tr}_R\, t^a t^b = C(R)\, \delta^{ab} \tag{5.28}$$

R ← Representation $\qquad \boxed{C(\text{fund}) = 1/2}$

The second Casimir operator of a group is defined by

$$t^a_R\, t^a_R = C_2(R)\, \mathbb{1}_R \tag{5.29}$$

(e.g. angular momentum squared L^2 in QM)

For the adjoint rep. we have

$$C_2(R) = C(R) = N \tag{5.30}$$

$$\Rightarrow (a) = \frac{1}{2} g^2 \mu^{2\varepsilon} N \int \frac{d^d l}{(2\pi)^d} \left[\delta_{\mu\rho} \cdots \right] \left[\delta_{\nu\rho} \cdots \right] \frac{1}{l^2} \frac{1}{(l+p)^2} \qquad (5.31)$$

We use the Feynman trick (QFT I, eq. 7.56, p.2.. and sheet 11)

$$\frac{1}{l^2} \frac{1}{(l+p)^2} = \int_0^1 d\alpha \, \frac{1}{\alpha (l+p)^2 + (1-\alpha) \, l^2}$$

$$= \int_0^1 d\alpha \, \frac{1}{(k^2 + \Delta)^2} \qquad (5.32)$$

with $k = l + \alpha p$ and $\Delta = \alpha(1-\alpha) \, p^2$.

We can also shift the l-integration: $\int d^d l = \int d^d k$.

It is left to rewrite the numerator in eq. (5.31) (see (5.26)) in terms of k: $l = k - \alpha p$.

(i) terms linear in k_μ integrate to zero:

$$\int d^d k \, \frac{1}{(k^2+\Delta^2)^2} \, k_\mu = 0 \qquad (5.33a)$$

(ii) terms with $k_\mu k_\nu$ are prop. to $\delta_{\mu\nu}$:

$$\int d^d k \, \frac{1}{(k^2+\Delta^2)^2} k_\mu k_\nu = \frac{1}{d} \int d^d k \, \frac{k^2}{(k^2+\Delta^2)^2} \qquad (5.33b)$$

In summary:

$$(a) = \delta^{ab} N g^2 \int_0^1 d\alpha\, \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\Delta)^2} \frac{N_{\mu\nu}}{2} \qquad (5.34)$$

with $N_{\mu\nu} = [\, \delta_{\mu\rho}(p-\ell)_\sigma + \delta_{\rho\sigma}(2\ell+p)_\mu + \delta_{\mu\sigma}(-2p-\ell)_\rho]$

$\cdot [\, \delta_{\nu\rho}(-p+\ell)_\sigma + \delta_{\rho\sigma}(-2\ell-p)_\nu + \delta_{\sigma\nu}(2p+\ell)_\rho]$

$\ell = k - \alpha p \rightarrow$ $= [\, \delta_{\mu\rho}(p(1+\alpha)-k)_\sigma + \delta_{\rho\sigma}(2k+p(1-2\alpha))_\mu + \delta_{\mu\sigma}(-k-(2-\alpha)p)_\rho]$

$\cdot [\, \delta_{\nu\rho}(-(1+\alpha)p+k)_\sigma + \delta_{\rho\sigma}(-2k-p(1-2\alpha))_\nu + \delta_{\nu\sigma}(k+(2-\alpha)p)_\rho]$

linear terms in k vanish

$$\boxed{= \delta_{\mu\nu}\left[-p^2\left((1+\alpha)^2+(2-\alpha)^2\right) - 2k^2\right] + p_\mu p_\nu\left((2-d)(1-2\alpha)^2 + 2(1+\alpha)(2-\alpha)\right) + k_\mu k_\nu (6-4d)\Big]} \qquad (5.35)$$

+ lin. terms

This amounts to $\left[\int d^d k\, k_\mu k_\nu f(k^2) = \frac{1}{d}\int d^d k\, k^2 f(k^2)\right]$

$$\boxed{(a) = \frac{\delta^{ab}}{2} N g^2 \int_0^1 d\alpha\, \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \Big[-\delta_{\mu\nu}\Big(p^2\left((1+\alpha)^2+(2-\alpha)^2\right) + 6k^2(1-1/d)\Big) + p_\mu p_\nu\left((2-d)(1-2\alpha)^2 + 2(1+\alpha)(2-\alpha)\right)\Big] \frac{1}{(k^2+\Delta)^2}} \qquad (5.36)$$

Now we use that

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \quad \frac{1}{\Delta^{n-d/2}} \quad \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

(see also sheet 11, QFT I) (5.37)

$$\Rightarrow \boxed{ (a) = \frac{1}{2} \frac{\delta^{ab} N g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}} \Big\{ -\delta_{\mu\nu}\, p^2 \Big[\Gamma(2-d/2)\big((1+\alpha)^2+(2-\alpha)^2\big) + \Gamma(1-d/2)\, 3\,(d-1)\,\alpha(1-\alpha) \Big] + p_\mu p_\nu \, \Gamma(2-d/2)\Big[(2-d)(1-2\alpha)^2 + 2(1+\alpha)(2-\alpha)\Big] \Big\} }$$

(5.38)

from eq. (3.37): $\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2-1}} \, \frac{d}{2}$

Tadpole (b):

$$= \underbrace{g^2 N(d-1)\delta^{ab}\delta_{\mu\nu}}_{\text{see sheet 9}} \mu^{2\varepsilon} \underbrace{\int \frac{d^d l}{(2\pi)^d} \, \frac{1}{l^2}}_{0 \text{ in dim-reg}}$$

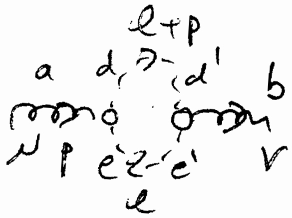
However, multiplying the integrand with $\frac{(p+l)^2}{(p+l)^2}$ leads to

$$= \frac{\delta^{ab}\delta_{\mu\nu} N g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}} (d-1)\, p^2 \Big\{ \Gamma(1-d/2)\, d\, \alpha(1-\alpha) + \Gamma(2-d/2)\, (1-\alpha)^2 \Big\}$$

(5.39)

Ghost-contribution ⓒ:

– fermion loop

$$\text{}$$


$$= -g^2\mu^{2\varepsilon}\int\frac{d^dl}{(2\pi)^d}\,\frac{1}{(l+p)^2}\,\frac{1}{l^2}\,\underbrace{f^{dae}f^{ebd}}_{-N}$$

$$\cdot\,(l+p)_\mu\, l_\nu \qquad (5.40)$$

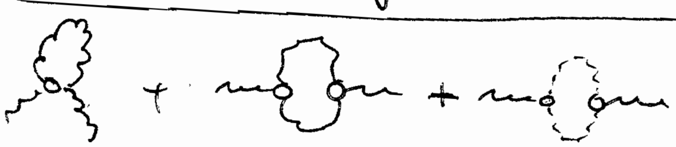
$$l = k-\alpha p \;\rightarrow\; = \delta^{ab} g^2 N\int_0^1 d\alpha\,\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2+\Delta)^2}\left(k_\mu k_\nu - \alpha(1-\alpha)p_\mu p_\nu\right)$$

$$= \frac{\delta^{ab}Ng^2}{(4\pi)^{d/2}}\int_0^1 d\alpha\,\frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}}\Big\{\delta_{\mu\nu}p^2\,\Gamma(1-d/2)\,\tfrac{1}{2}\alpha(1-\alpha)$$

$$-\,p_\mu p_\nu\,\Gamma(2-d/2)\,\alpha(1-\alpha)\Big\}$$

$$(5.41)$$

Pure Yang-Mills:



$$= -\frac{\delta^{ab}Ng^2}{(4\pi)^{d/2}}\int_0^1 d\alpha\,\frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}}\,\Gamma(2-d/2)$$

$$\cdot\left(\delta_{\mu\nu}p^2 - p_\mu p_\nu\right)\left\{\left(1-\frac{d}{2}\right)(1-2\alpha)^2+2\right\}$$

$$(5.42)$$

For $\varepsilon \to 0$ the dimension d approaches 4: $d = 4-2\varepsilon$

Then, eq. (3.34) diverges as $1/\varepsilon$ because of $\Gamma[2-d/2] = \Gamma[\varepsilon]$, see also QFT I, chapter 7, p. 187:

$$\Gamma[\varepsilon] = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)$$

Euler-Mascheroni $\gamma = 0.577\ldots$

Hence in this limit we get

$$= -\frac{g^2 N}{16\pi^2}\left(p^2 \delta_{\mu\nu} - p_\mu p_\nu\right)\delta^{ab} \cdot \Big[\frac{5}{3}\left(\frac{1}{\varepsilon} - \ln p^2/\mu^2\right) + \text{finite}\Big] \quad (5.43)$$

$$\left[\int_0^1 d\alpha \,(1-d/2)(1-2\alpha)^2 + 2\right]$$

Remarks: (1) Singular diagrams are always proportional to $(1/\varepsilon - \ln p^2/\mu^2)$.

$$\frac{\mu^{2\varepsilon}}{(\alpha(1-\alpha)p^2)^\varepsilon} = 1 + \varepsilon \ln \mu^2/p^2 - \varepsilon \ln \alpha(1-\alpha) + O(\varepsilon^2)$$

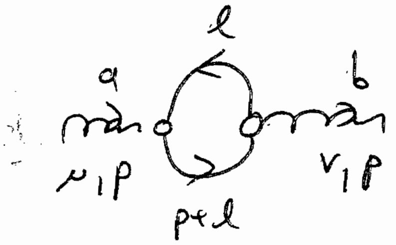
(2) For general $\xi \neq 1$ we get

$$\tfrac{5}{3} \rightarrow \tfrac{5}{3} + \tfrac{1}{2}(1-\xi) \qquad (5.44)$$

Exercise!

In the presence of quarks we have to add the diagram (d):

Quark contribution (d):



$$= \underset{\substack{\Downarrow \\ \text{fermion loop}}}{-}\, g^2 \mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} (-1)\, \overset{\substack{i^2 \\ \Downarrow}}{} \operatorname{tr}_{\text{fund}} \underbrace{t^a t^b}_{\substack{\| \\ \frac{1}{2}\delta^{ab}}} \qquad (5.45)$$

$$\operatorname{tr}_{\text{Dirac}} \cdot \frac{1}{i\not{l}+m}\, \gamma_\nu\, \frac{1}{i(\not{l}+\not{p})+m}\, \gamma_\mu$$

[Computation see QFT I, chapter 7.2, p. 200-204]

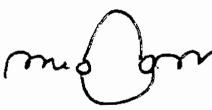
$\Rightarrow$ 

$$= -\frac{1}{2} \frac{g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{\mu^{2\varepsilon}}{(\Delta+m^2)^{2-d/2}} \frac{4}{d-1} \frac{1}{p^2}$$

$$\Big\{ (d-2)\,\Gamma(1-d/2)(\Delta+m^2) - (d-2)\,\Gamma(2-d/2)(2\Delta+m^2) + d\,\Gamma(2-d/2)\, m^2 \Big\}$$

$$\cdot\, \delta^{ab}\left(p^2 \delta_{\mu\nu} - p_\mu p_\nu\right) \qquad (5.46)$$

$d \to 4$:

 $$= \frac{4g^2}{(4\pi)^2} \int_0^1 d\alpha \, \alpha(1-\alpha) \frac{\Gamma(2-d/2)}{(\Delta + m^2)^{\varepsilon}} \cdot \delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \qquad (5.45)$$

$$+ \text{finite}$$

$$= \frac{g^2 N}{16\pi^2} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \delta^{ab} \cdot \left[\frac{2}{3} \cdot \left(\frac{1}{\varepsilon} - \int_0^1 d\alpha \ln \frac{\Delta + m^2}{\mu^2} \right) + \text{finite} \right]$$

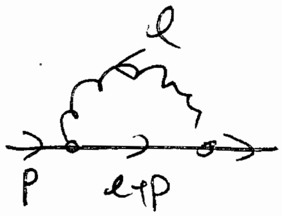
In summary we get for the one-loop correction

for  $^{-1}\big|_{1\text{-loop}} = \frac{\delta^2 \Gamma}{\delta A^2}$: $\boxed{m = 0}$

 $$^{-1}\Big|_{1\text{-loop}} = \frac{g^2 N}{16\pi^2} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \delta^{ab} \cdot \left[\left(\frac{5}{3} + \frac{1}{2}(1-\xi) \right) N - \frac{2}{3} \right] \left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right)$$

$$+ \text{finite} \qquad (5.46)$$

For the β-function we also need the quark self-energy: $\boxed{\xi = 1}$



$$= -g^2 \mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} t^a \gamma_\nu \frac{1}{i(\slashed{l}+\slashed{p})} \gamma_\nu t^a$$

$t^a_f t^a_f$ $\longrightarrow$

$= C_2(\text{fund})\, \mathbb{1}_f$

p. 117

$C_2(\text{fund}) = \frac{N^2-1}{2N}$

$\text{tr}\, t^a t^a = C_2(R) \cdot \dim(R)$

$= C(R) \cdot (N^2-1)$

$$= \frac{(N^2-1)}{2N} \frac{g^2}{(4\pi)^{d/2}} i \slashed{p} \mu^{2\varepsilon} \int_0^1 d\alpha (1-\alpha)(d-2) \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

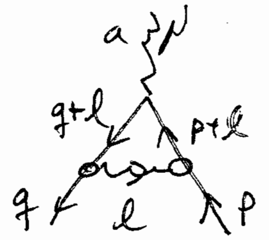
$$= \frac{N^2-1}{2N} \frac{g^2}{(4\pi)^{d/2}} i \slashed{p} \left[\left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right) + \text{finite} \right] \qquad (5.47)$$

Gauge dependence: Exercise

$$\left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right) \rightarrow \xi \left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right) \qquad (5.48)$$

Accordingly, there is no singularity in the self-energy for Landau gauge: $\xi = 1$.

Quark - gluon vertex:



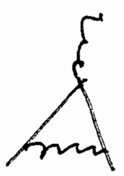
$$= i g^3 \int \frac{d^d l}{(2\pi)^4} \frac{t^b t^a t^b \, \gamma_\nu (\not{l}+\not{q}) \gamma_\mu (\not{l}+\not{p}) \gamma_\nu}{l^2 \, (l+q)^2 \, (l+p)^2} \tag{5.49}$$

with $t^b t^a t^b$

$$= t^{b2} \, t^a + t^b [t^a, t^b]$$

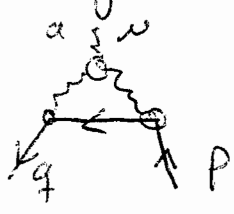
$$= C_2(\text{fund}) t^a + i \, t^b f^{abc} t^c$$

$$= \left[C_2(\text{fund}) - \frac{1}{2} C_2(\text{adj.}) \right] t^a$$

$\Rightarrow$  $\Big|_{q,p=0}$

$$\simeq i g t^a \frac{g^2}{(4\pi)^{d/2}} \left(\frac{1}{2} \, \frac{N^2-1}{N} - \frac{N}{2} \right) \cdot \Big(\underbrace{\Gamma\left(2 - \tfrac{d}{2}\right)}_{\xrightarrow{d\to4} 1/\varepsilon} + \text{finite} \Big) \tag{5.50}$$

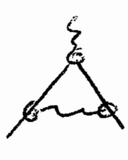
Similarly:



$$\simeq i g t^a \frac{g^2}{(4\pi)^{d/2}} \, \frac{3}{2} N \left(\Gamma\left(2 - d/2\right) + \text{finite} \right)$$

$d \to 4$

and hence

 + 

$$\simeq i g t^a \frac{g^2}{(4\pi)^{d/2}} \Big(\overbrace{\tfrac{3}{2} N - \frac{1}{2N}}^{C_2(\text{fund}) + C_2(\text{adj.})} \Big) \left(\Gamma(2 - d/2) + \text{finite} \right) \tag{5.51}$$

β-function: We recall that

$$\beta_g = -g\,\mu \frac{d}{d\mu} \ln Z_g$$

and $Z_g = Z_{1,F} / Z_2 Z_3^{1/2} = \dfrac{Z_{1,F}}{Z_\psi Z_A^{1/2}}$ (5.52)

We write $Z = 1 + \delta Z$ ← cancels infinity and get at one loop:

$$Z_g = 1 + \delta Z_{1,F} - \delta Z_\psi - \frac{1}{2}\delta Z_A \quad (5.53)$$

We collect the δZ's:

N_f flavour

$$\delta Z_A \simeq \frac{g^2 N}{16\pi^2}\left[\left(\frac{5}{3} + \frac{1}{2}(1-\xi)\right) - \frac{2}{3}N_f\right]\frac{1}{\varepsilon}$$

$$\delta Z_\psi \simeq -\frac{g^2}{16\pi^2}\,\frac{N^2-1}{2N}\,\xi\,\frac{1}{\varepsilon}$$

$$\delta Z_{1,F} \simeq -\frac{g^2}{16\pi^2}\left[\left(1 - \frac{1-\xi}{4}\right)N + \xi\,\frac{N^2-1}{2}\right]\frac{1}{\varepsilon}$$

(5.54)

We also have

$$\delta Z_c = \frac{g^2 N}{16\pi^2}\left[\frac{1}{2} + \frac{1-\xi}{4}\right]\frac{1}{\varepsilon}$$

(5.55)

and

$$\underset{(\bar{c}Ac)}{\delta \tilde{Z}_1} = -\frac{g^2 N}{16\pi^2}\,\xi/2\,\frac{1}{\varepsilon}$$

$\Rightarrow$

$$\beta(g) = -g\,\mu \frac{d}{d\mu} \ln Z_g$$

$$= -g\,\mu \frac{d}{d\mu}\left(\delta Z_{1,F} - \delta Z_\psi - \frac{1}{2}\delta Z_A\right) + O(2\text{-Loop})$$

$\delta Z \sim \left(\frac{1}{\varepsilon} - \ln \mu^2/\bar{\mu}^2\right)$

$$= \frac{g^3}{16\pi^2}\,2\Big[\overbrace{-\left(\left(1-\frac{1-\xi}{4}\right)N - \xi\frac{N^2-1}{2N}\right)}^{\delta Z_{1,F}} \qquad (5.56)$$

$$+ \underbrace{\frac{N^2-1}{2N}\xi}_{-\delta Z_\psi} \underbrace{-\frac{1}{2}\left(\frac{5}{3} + \frac{1}{2}(1-\xi)\right)N - \frac{2}{3}N_f}_{1/2\,\delta Z_A}\Big]$$

$$\Rightarrow \boxed{\begin{aligned} \beta(g)\Big|_{1\text{-loop}} &= \frac{g^3}{16\pi^2}\left[-\frac{11}{3}\underset{\substack{\uparrow\\ C_2(\mathrm{adj})}}{N} + \frac{4}{3}N_f\,\underset{\substack{\uparrow\\ C(\mathrm{fund})}}{\frac{1}{2}}\right] \\ &= \beta_1 \cdot g \end{aligned}} \qquad (5.57)$$

Remarks:

(1) For $N_f < \frac{22}{4} N$: $\beta < 0$

(2) The β-function can be integrated to obtain

$$\alpha_s(p) = \frac{\alpha_s^0}{1 - \beta_1 \ln p^2/\mu_0^2} \qquad (5.58)$$

$p \to \infty$: $\alpha_s(p) \simeq -\dfrac{\alpha_s^0}{\beta_1 \ln p^2/\mu_0^2} \to 0$ asymptotic freedom

Remarks:

(1) In $SU(N)$ the property eq. (6.42) relevant for gauge invariance requires

$$\mathcal{U}_{\mathcal{C}_{x,y}} = \mathcal{P}\, e^{ig \int_{\mathcal{C}_{x,y}} dz_\nu A_\nu(z)} \qquad (6.44)$$

path ordering (remember time-ord.)

(2) The covariant derivative follows as

$$\mathcal{U}^{\dagger}_{\mathcal{C}_{y,x}} \partial_\nu \mathcal{U}_{\mathcal{C}_{y,x}} = \partial_\nu + ig A_\nu \qquad (6.45)$$

This property can be used to write the Dirac eq. in terms of a phase factor $W_{\mathcal{C}}$ and the free Dirac eq..

In summary we conclude that the expectation value of a static $q\bar{q}$-pair is given by

$$\boxed{W[L,T] = \frac{1}{Z}\int dA\; W_{\mathcal{C}}(A)\, e^{-S_{YM}[A]}} \qquad (6.46)$$

for $\mathcal{U}_{\mathcal{C}}$ see p. 141