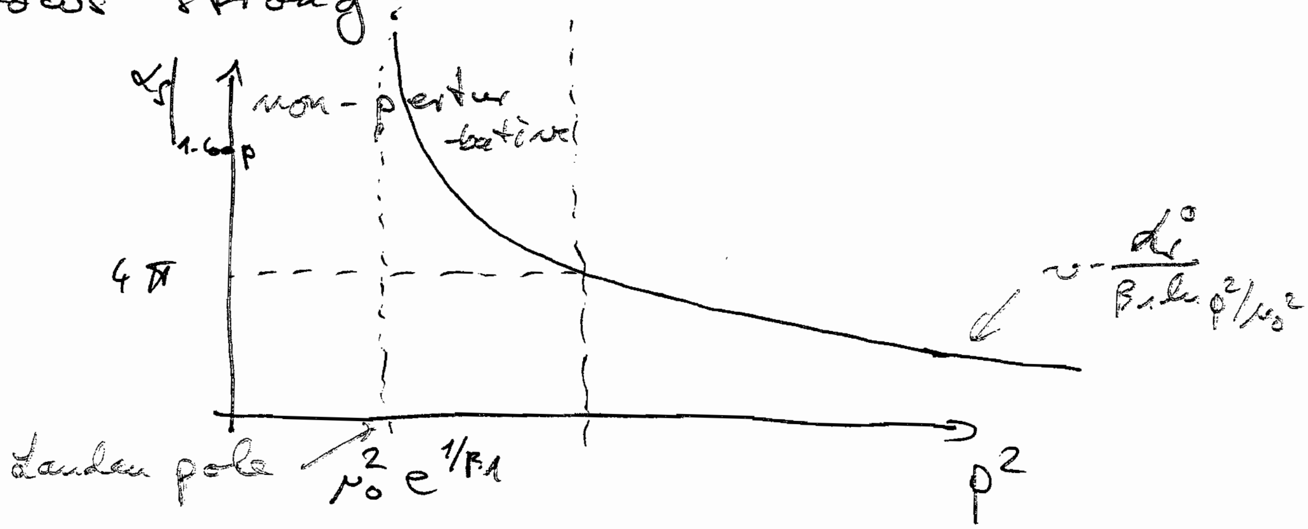
6 Lattice gauge theory

The QCD coupling α_s tends to zero for large momenta, see eq. (5.58), and perturbation theory works.

For momentum p getting small, the coupling grows strong:



Hence, for low momenta <u>non</u>-perturbative methods are required: $\boxed{\text{Numerically solving the path-int.}}$

Remark: Taking into account higher orders in perturbation theory only slightly shifts the Landau pole, and does not remove it.

6.1 Scalar fields on the lattice

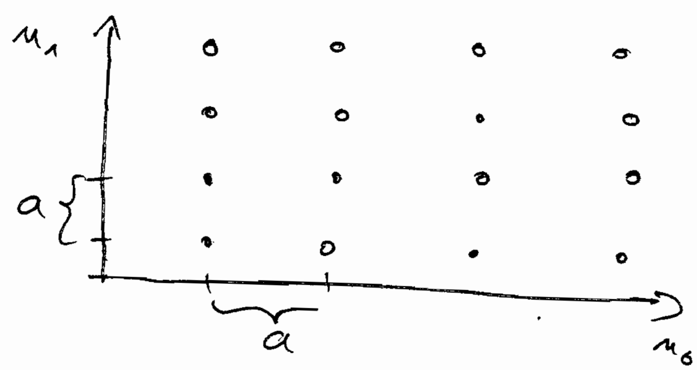
We consider the action of a complex scalar field,

$$S[\phi] = \int d^4x \, \phi^\dagger(x) \left(-\underbrace{\partial_\nu \partial_\nu}_{\Delta} + m^2 \right) \phi(x) \quad (6.1)$$

In chapter 1.2 we derived the path integral from a discretised integral in time-direction. Now consider a discretised integral in all directions:

$$\mathcal{D}\phi(x) = \prod_{n} d\phi(na) \qquad (6.2)$$

with $n = \begin{pmatrix} n_0 \\ \vdots \\ n_3 \end{pmatrix}$, $n_\mu \in \mathbb{Z}$



with a hypercubic lattice with <u>lattice spacing</u> a.

We have the substitutions

$$x_\mu \to n_\mu a$$

$$\phi(x) \to \frac{1}{a}\hat{\phi}_n$$

$$\int d^4x \to a^4 \sum_{\hat{n}} \qquad (6.3)$$

$$\Delta\phi \to \frac{1}{a^2}\hat{\Delta}\phi$$

$$m \to \frac{1}{a}\hat{m}$$

dimensionless

$\hat{\mu} = e_\mu$: unit vector in μ-dir.

with
$$\hat{\Delta}\hat{\phi}_n = \sum_\mu \left[\hat{\phi}_{n+\hat{\mu}} + \hat{\phi}_{n-\hat{\mu}} - 2\hat{\phi}_n\right] \qquad (6.4)$$

discrete Laplacian

Note that
$$\hat{\Delta}\phi(\hat{n}a) = \sum_\mu \Big[\underbrace{(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_n)}_{\hat{\partial}^R_\mu \hat{\phi}_n} - \underbrace{(\hat{\phi}_n - \hat{\phi}_{n-\hat{\mu}})}_{\hat{\partial}^L_\mu \hat{\phi}_n}\Big]$$

and $\hat{\partial}_\mu \hat{\phi} = \frac{1}{2}\left(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_{n-\hat{\mu}}\right)$ ← symmetric deriv. $\qquad (6.5)$

The derivatives $\frac{1}{a}\hat{\Delta}$, $\frac{1}{a}\hat{\partial}_{L/R_\mu} \xrightarrow{a\to 0} \Delta, \partial_\mu$ in the cont. limit.

In summary, the discretised lattice action of a scalar field takes the form

$$S[\hat{\phi}] = \sum_{n,m} \phi_n^\dagger K_{nm} \hat{\phi}_m \qquad (6.6)$$

with
$$K_{nm} = -\sum_{\mu>0}\left[\delta_{\hat{n}+\hat{\mu},\hat{m}} + \delta_{\hat{n}-\hat{\mu},\hat{m}} - 2\delta_{\hat{n},\hat{m}}\right] + \hat{m}^2 \delta_{nm} \qquad (6.7)$$

This leads us to the generating functional of the free scalar lattice theory:

$$Z_0[\hat{J}] = \int \prod_n d\hat{\phi}_n \, e^{-S[\hat{\phi}] + \sum_n \hat{J}_n \hat{\phi}_n + \sum_n \hat{J}_n^+ \hat{\phi}_n^+}$$

$$= \frac{1}{\sqrt{\det K}} \, e^{\sum_{n,m} \hat{J}_n^+ (K^{-1})_{nm} \hat{J}_m} \qquad (6.8)$$

From eq. (6.8) we derive the propagator

$$\langle \hat{\phi}_n^+ \hat{\phi}_m \rangle = K^{-1}_{nm} \qquad (6.9)$$

with

$$K_{nm} \cdot K^{-1}_{mn'} = \delta_{nn'} \qquad (6.10)$$

Momentum space representation:

$$K_{nm} = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} \tilde{K}(\hat{p}) \, e^{i\hat{p}(n-m)} \qquad (6.11)$$

with $\delta_{nm} = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} e^{i\hat{p}(n-m)}$

and

$$\boxed{\tilde{K}(\hat{p}) = 4 \sum_{\mu=0}^{3} \sin^2 \frac{\hat{p}_\mu}{2} + \hat{m}^2} \qquad (6.12)$$

Note that $\sum_n e^{i\hat{p}n} = (2\pi)^4 \delta^4(\hat{p})$

The momentum integration in eq. (6.12) is restricted to the <u>Brillouin zone</u> $\hat{p}_\mu \in [-\pi, \pi]$.

Continuum limit:

$$(i)\quad \lim_{a\to 0} \frac{1}{a^2}\,\tilde{K}(\hat{p}) = \frac{1}{a^2}\left[4\sum_{\mu=0}^{3}\left(\frac{1}{4}(a p_\mu)^2 + \mathcal{O}(a^4)\right) + a^2 m^2\right]$$

$$= p^2 + m^2 \qquad (6.13)$$

$$(ii)\quad \lim_{a\to 0} 1/a^2 \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4}\, \frac{e^{i\hat{p}(n-m)}}{\tilde{K}(\hat{p})} = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4}\, \frac{e^{ip(\overset{na}{x} - \overset{m\cdot a}{y})}}{p^2+m^2} \qquad (6.14)$$

$$(iii)\quad S[\hat{\phi}] \rightarrow S[\phi]$$

Remarks:

(1) The discrete lattice generating functional can be computed numerically:

(a) put the theory on a finite lattice: $|n_\mu a| \leq L$

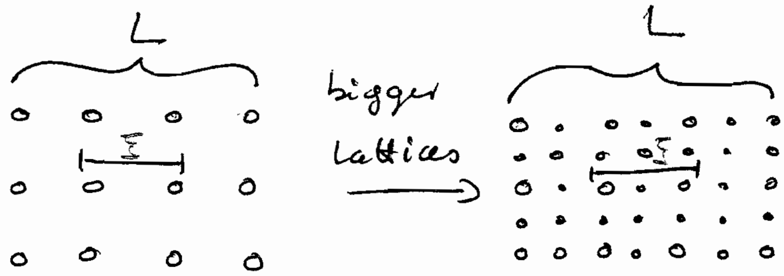
(b) Perform $(L/a)^4$ integrations with Monte-Carlo methods. Common sizes for dynamical QCD simulations $\sim 16^4, 32^4$.

(c) Interaction term

$$S_I(\hat{\phi}) = \frac{\hat{\lambda}}{4!} \sum_n \hat{\phi}^4(\hat{n} a) \qquad (6.15)$$

(2) Continuum limit:

(a)



ξ: correlation length

(b)