6.2 Non-Abelian gauge fields on the lattice

As in the continuum we introduce gauge trafo.. for the complex matter field $\hat{\Phi}$ (fund. rep.

$$\hat{\Phi}_n \to G(n)\,\hat{\Phi}_n \quad \text{with } G(n) \in SU(N) \tag{6.16}$$

Looking at the action eq. (6.6) on p. 131 we have to ensure gauge invariance for

e.g. $$\hat{\Phi}^{+}_{n}\,\hat{\Phi}_{n+\hat{\mu}} \tag{6.17}$$

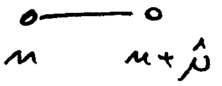
in the action $S[\hat{\Phi}] = \frac{1}{2}\sum_{n,m} \hat{\Phi}^{+}_{n} K_{nm}\,\hat{\Phi}_{m}$.

This entails that we have to <u>transport</u> the group element $G^{+}(n+\hat{\mu})$ from the lattice point $n+\hat{\mu}$ to n: We define the link variable

$$U_\mu(n) : U_\mu(n) \to G(n)\,U_\mu(n)\,G^{+}(n+\hat{\mu}) \tag{6.18}$$

with parameterisation $\underset{\in\, SU(N)}{U_\mu(n)} = e^{i\,\hat{\varphi}(n)}$ $\;\hat{\varphi} \in su(N)$

(1) $U_\mu(n)$ 'lives' on the link between n and $n+\hat\mu$:



(2) The term

$$\hat\phi^\dagger_n \; U_\mu(n)\; \hat\phi_{n+\mu} \to \hat\phi^\dagger_n G^\dagger(n) G(n) U_\mu(n) G^\dagger(n+\hat\mu) G(n+\hat\mu)\, \phi_n$$
$$= \hat\phi^\dagger_n \; U_\mu(n)\; \hat\phi_{n+\mu} \qquad (6.19)$$

is gauge invariant. This also holds for $\phi^\dagger_n U^\dagger_\mu(n-\hat\mu)\phi_{n-\mu}$. ... Then the action turns into (6.20)

$$S[\hat\phi, U] = -\sum_{\substack{n\\ \mu>0}} \left(\hat\phi^\dagger_n U^\dagger_\mu(n-\hat\mu)\phi_{n-\hat\mu} + \phi^\dagger_n U_\mu(n)\hat\phi_{n+\hat\mu} \right) + \sum_n \phi^\dagger_n \phi_n (\hat m^2 + 8)$$

(3) Continuum limit: Write $U_\mu(n) = e^{i \boxed{g_0} a A_\mu(n)}$ (6.21)

bare lattice coupling

and expand (6.20) in powers of the lattice spacing a.

$$\boxed{\hat\varphi(n) = i g_0 a A_\mu(n)} \qquad (6.22)$$

We get

$$S[\hat\phi, U] \xrightarrow{a\to 0} -\sum_{n,m} \hat\phi_n^+ K_{nm} \hat\phi - \sum_{\substack{n\\ \mu>0}} \hat\phi_n^+ \Big[-iag_0 A_\mu(n-\hat\mu)\hat\phi_{n-\hat\mu} + iag_0 A_\mu(n)\hat\phi_{n+\hat\mu}\Big]$$

$$O(a^2) + iag_0 \Big[\underbrace{\big(A_\mu(n) - A_\mu(n-\hat\mu)\big)}_{\hat\partial_\mu^L A_\mu(n)} \hat\phi_n + A_\mu(n) \underbrace{\big(\hat\phi_{n+\hat\mu} - \hat\phi_{n-\hat\mu}\big)}_{\hat\partial_\mu \hat\phi} \Big]$$

eqs. (6.3), 6.5))

$$+ \sum_n \hat\phi_n^+ \hat\phi_n \hat m^2 - \sum_n \hat\phi_n^+ (iag_0 A_\mu)^2 \hat\phi_n$$

$$\to -\int d^4x \; \phi^+(x) \, D_\mu^2 \, \phi(x) \qquad (6.23)$$

Remarks: (1) the definition $\hat\rho(n) = iag_0 A_\mu(n)$

is only 'unique' up to order a^2.

(2) Field strength $F_{\mu\nu}$:

(a) in continuum:

curvature $\frac{1}{ig}[D_\mu, D_\nu]$

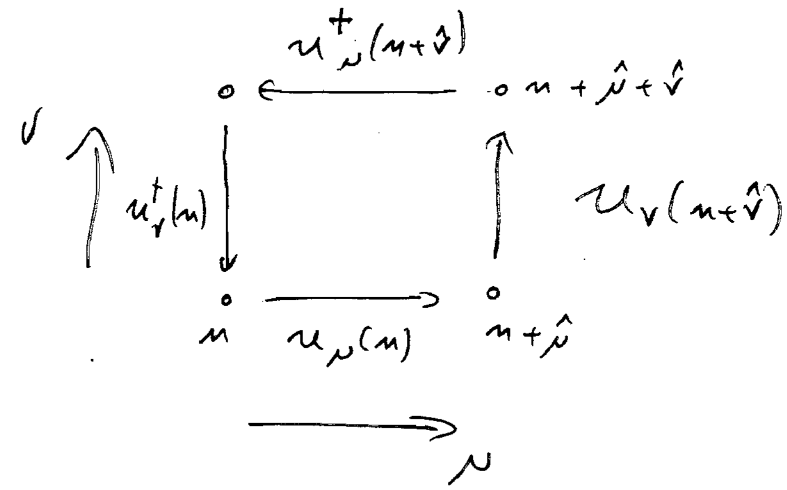
(b) 'Curvature' on the lattice

$$\boxed{U_{\mu\nu}(n) = U_\mu(n)\, U_\nu(n+\hat\mu)\, \hat U_\mu^+(n+\hat\nu)\, U_\nu^+(n)} \qquad (6.24)$$

$\| $

U_p Plaquette variable

$U_{\mu\nu}(n)$:



We define in analogy

$$U_{\mu\nu}(n) = e^{i g_0 a^2 F_{\mu\nu}(n)} \qquad (6.25)$$

with lattice fieldstrength tensor $F_{\mu\nu}(n)$.

In QED we derive from eq. (6.24),

$$F_{\mu\nu}(n) = \frac{1}{a}\Big[\overbrace{\big(A_\nu(n+\hat{\mu}) - A_\nu(n)\big)}^{\hat{\partial}^R_\mu A_\nu} - \overbrace{\big(A_\mu(n+\hat{\nu}) - A_\mu(n)\big)}^{\hat{\partial}^R_\nu A_\mu}\Big] \qquad (6.26)$$

In QCD we can use Baker-Campbell-Hausdorff

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \qquad (6.27)$$

and arrive at

$$F_{\mu\nu}(n) = \frac{1}{a}\Big[\hat{\partial}^R_\mu A_\nu - \hat{\partial}^R_\nu A_\mu + i g_0 a [A_\mu, A_\nu](n) + \mathcal{O}(a)\Big] \qquad (6.28)$$

This allows us to derive an lattice analogue of the pure Yang-Mills action,

$$S_W[\mathcal{U}_\rho] = \beta \sum_{\substack{n \\ \mu<\nu}} \left(1 - \frac{1}{2N} \mathrm{tr}_{\mathrm{fud}}\left(\mathcal{U}_{\mu\nu}(n) + \mathcal{U}^\dagger_{\mu\nu}(n)\right)\right) \tag{6.28}$$

with

$$\beta = 2N/g_0^2 \tag{6.29}$$

This follows straightforwardly from

$$\mathcal{U}_{\mu\nu}(n) \xrightarrow{a\to 0} 1 + i g_0 a\, \tilde{\mathcal{F}}_{\mu\nu}(n) - \frac{g_0^2 a^2}{2} \tilde{\mathcal{F}}^2_{\mu\nu}(n) + \mathcal{O}(a^3) \tag{6.30}$$

and hence

$$\mathcal{U}_{\mu\nu}(n) + \mathcal{U}^\dagger_{\mu\nu}(n) = -g_0^2 a^2\, \underset{\text{no. sum}}{\tilde{\mathcal{F}}^2_{\mu\nu}}(n) + \mathcal{O}(a^3) \tag{6.31}$$

Inserting eq. (6.31) in (6.28) we are led to

$$S_W[\mathcal{U}_\rho] \xrightarrow{a\to 0} S_{YM}[A] = \frac{1}{2}\int d^4x\, \mathrm{tr}_{\mathrm{fud}} F^2_{\mu\nu} \tag{6.32}$$

$$\sum_{\mu<\nu} \left(1 - \frac{1}{2N} \mathrm{tr}\left(\mathcal{U}_{\mu\nu} + \mathcal{U}^\dagger_{\mu\nu}\right)\right) = \sum_{\mu<\nu} \frac{g_0^2 a^2}{2N} \mathrm{tr}\, \tilde{\mathcal{F}}^2_{\mu\nu} + \mathcal{O}(a^3)$$

$$= g_0 a^2/_{4N}\ \mathrm{tr}\, \tilde{\mathcal{F}}^2_{\mu\nu} \ldots + \mathcal{O}(a^3).$$

Lattice generating functional for YM-theory:

$$Z = \int \mathcal{D}U \, e^{-S_W[U_\mu]} \qquad (6.33)$$

and

$$\langle U_{\mu_1}^{a_1 b_1}(n_1) \cdots U_{\mu_m}^{a_m b_m}(n_m) \rangle$$

$$= \frac{1}{Z} \int \mathcal{D}U \, U_{\mu_1}^{a_1 b_1}(n_1) \cdots U_{\mu_m}^{a_m b_m}(n_m) e^{-S_W[U_\mu]} \qquad (6.34)$$

where

$$\mathcal{D}U = \prod_{\underset{\uparrow \text{links}}{l}} \underset{\nwarrow \text{Haar measure}}{dU_l}$$

with

SU(3):

$$\int dU \, U^{ab} = 0$$

$$\int dU \, U^{ab} U^{cd} = 0 \qquad (6.35)$$

$$\int dU \, U^{ab} (U^+)^{cd} = \frac{1}{3} \delta_{ad} \delta_{bc}$$

$$\int dU \, U^{a_1 b_1} U^{a_2 b_2} U^{a_3 b_3} = \frac{1}{3!} \varepsilon_{a_1 a_2 a_3} \varepsilon_{b_1 b_2 b_3}$$

and

$$\underset{\underset{VUV^+}{\|}}{\int dU'} = \int dU = \underset{\underset{\int dU \, \underbrace{U^{ab}(U^+)^{bd}}_{\delta_{ad}} = \delta_{ad}}{\uparrow}}{1} \qquad (6.36)$$