6.5 Fermions on the lattice

We conclude our lattice excursion with a few remarks on fermions. We put fermions on the lattice similarly to scalars:

$$S_D[\psi, \bar{\psi}] = -\int d^4x\, \bar{\psi}(\not{\partial} + m)\psi \qquad (6.80)$$

see eq. (2.30), p. 48. The lattice substitutions are analoguously to eq. (6.3) for scalars:

$$\begin{aligned} \overset{(-)}{\psi}(x) &\longrightarrow \frac{1}{a^{3/2}}\, \overset{(-)}{\hat{\psi}}(n) \\ \partial_\mu \psi(x) &\longrightarrow \frac{1}{a^{5/2}}\, \hat{\partial}_\mu \hat{\psi}_n \\ m &\longrightarrow 1/a\, \hat{m} \end{aligned} \qquad (6.81)$$

with

$$\hat{\partial}_\mu \hat{\psi}(n) = \frac{1}{2}\left(\hat{\psi}_{n+\hat{\mu}} - \hat{\psi}_{n-\hat{\mu}}\right)$$

being the symmetric lattice derivative, see eq. (6.5).

Then, the lattice action reads

$$S_D = \sum_{\substack{n,m \\ \alpha,\beta}} \bar{\hat{\psi}}_{\alpha,n} \, K_{\alpha\beta,nm} \, \hat{\psi}_{\beta,m} \qquad (6.82)$$

spinor indices

with

$$K_{\alpha\beta,nm} = -\sum_\nu \frac{1}{2}(\gamma_\nu)_{\alpha\beta}\left[\delta_{m,n+\hat{\nu}} - \delta_{m,n-\hat{\nu}}\right] + \hat{m}\,\delta_{mn}\,\delta_{\alpha\beta} \qquad (6.83)$$

and the generating functional is given by

$$Z[\eta,\bar{\eta}] \simeq \int \prod_{n,\alpha} d\hat{\psi}_{\alpha,n} \prod_{m,\beta} d\bar{\hat{\psi}}_{\beta,m}\, e^{-S_D + \sum_{n,\alpha}\left(\bar{\eta}_{n,\alpha}\hat{\psi}_{\alpha,n} - \bar{\hat{\psi}}_{\alpha,n}\eta_{n,\alpha}\right)} \qquad (6.84)$$

The Grassmann integrals in eq. (6.84) are performed straight forwardly and yield

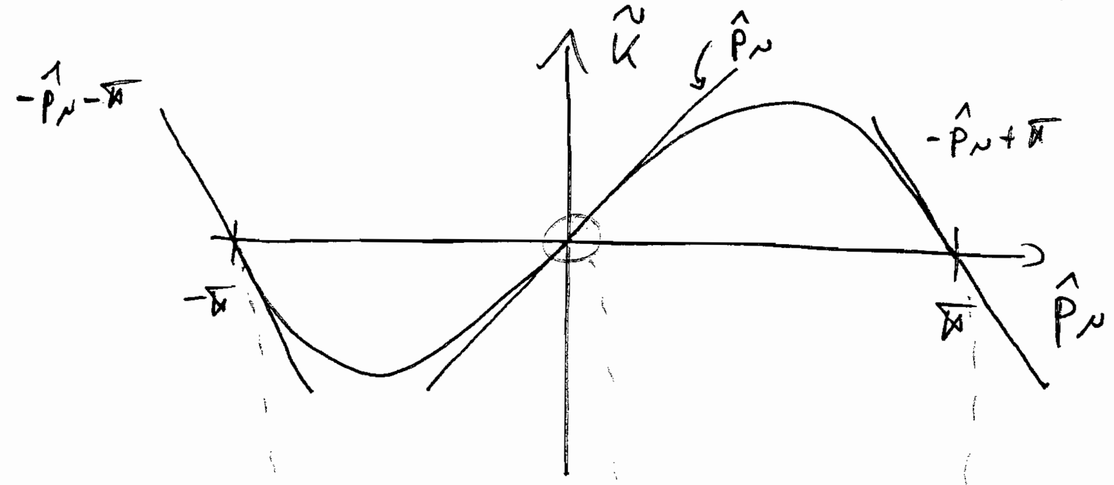
$$\boxed{Z[\eta,\bar{\eta}] \simeq \det K \cdot e^{-\sum_{n,m,\alpha,\beta} \bar{\eta}_{\alpha,n} K^{-1}_{\alpha\beta,nm}\eta_{\beta,m}} \qquad (6.85)}$$

It is left to invert K. As in the scalar case we go to momentum space:

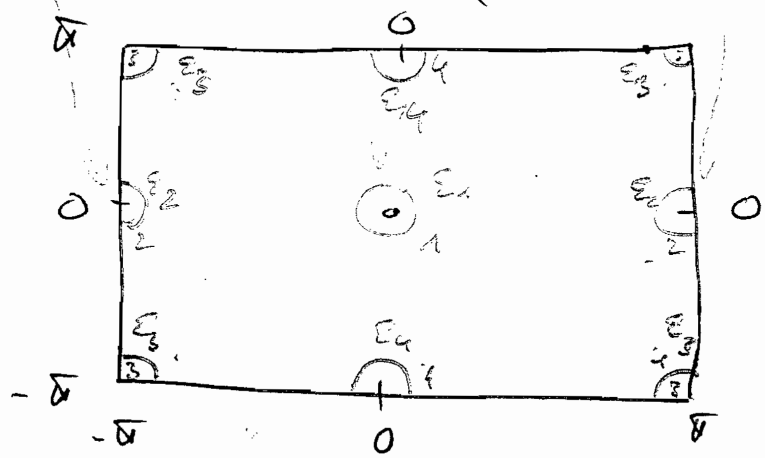
$$K_{\alpha\beta, nm} = \int_{-\pi}^{\pi} \frac{d^4 \hat{p}}{(2\pi)^4} \tilde{K}_{\alpha\beta}(\hat{p})\, e^{i\hat{p}(n-m)} \tag{6.86}$$

with
$$\tilde{K}_{\alpha\beta}(\hat{p}) = -\sum (\gamma_\nu)_{\alpha\beta} \sin \hat{p}_\nu - \hat{m} \tag{6.87}$$

$\Rightarrow$ Dispersion relation: (all momenta $\hat{p}_\nu = 0$ for $\nu \neq \mu$)



Zeros in two dim: (Brillouin zone)



$\Rightarrow$ 4 fermions

in d dim

$\boxed{2^d \text{ fermions}}$

ε_i: neighbourhood of zeros

Continuum limit for propagator: (see eq. (6.14))

$$\lim_{a\to 0} \frac{1}{a^3} \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} \frac{e^{i\hat{p}(n-m)}}{\tilde{K}(\hat{p})}$$

$$= - \lim_{a\to 0} \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\gamma_\nu \frac{1}{a} \sin a p_\nu + m} \tag{6.88}$$

$$= - \sum_i \int_{\varepsilon_i} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{(-1)^{i+1} \gamma_\nu p_\nu + m} + \mathcal{O}(a)$$

Note that the sign of the γ-matrices can be flipped by a $\hat{p}$-dependent similarity transformation:

$$\Psi_{\hat{p}} \gamma_\nu \Psi_{\hat{p}}^{-1} = (-1)^{i+1} \gamma_\nu \tag{6.89}$$

In summary we have produced 16 fermions on the lattice. This is the (in)famous doubling problem on the lattice.

Remarks:

(1) For massive fermions we can suppress the 15 doublers with ($\hat{\Delta}$ in (6.4), (6.7))

$$\frac{r}{2} \sum_{\substack{n,m \\ \alpha,\beta}} \bar{\hat{\psi}}_{\alpha,n} \hat{\Delta}_{n,m} \hat{\psi}_{\beta,m} \qquad (6.90)$$

Wilson term $\quad \hat{\Delta}_{n,m} = \sum_{\mu>0} [\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}]$

It diverges like $\frac{1}{a}$ $\quad \varepsilon_i, i \neq 1$, it vanishes with a in ε_1. [No chiral sym.]

(2) This does not work for <u>chiral</u> fermions: Nielsen-Ninomiya no-go theorem

'One cannot put <u>one</u> <u>Weyl</u> fermion on the lattice'

Ways out:

(a) break chiral sym. a la (6.90) on the lattice: numerical costs high for recovering chiral sym. in cont. limit

(b) maximal chiral invariance: $S_D = \sum_{\substack{n,m \\ \alpha,\beta}} \bar{\hat{\psi}}_{\alpha,n} \mathcal{D}_{\alpha\beta,nm} \cdot \hat{\psi}_{\beta,m}$

Ginsparg-Wilson relation (RG-arguments)

Consider Dirac operators $\mathcal{D}$ on the lattice

with

$$\boxed{\gamma_5 \mathcal{D} + \mathcal{D} \gamma_5 = 2a\, \mathcal{D} \gamma_5 \mathcal{D}} \qquad (6.91)$$

Eq. (6.91) allows the definition of -deformed- chiral transformations

$$\psi \to e^{i\alpha \tilde{\gamma}_5} \psi \qquad (6.92)$$

with $\tilde{\gamma}_5 = (1 - a\mathcal{D}) \gamma_5$. Exercise

Numerical costs: solutions of eq. (6.91) involve roots → high numerical costs

(c) staggered fermions (a standard choice)

look up books for details

Numerical costs: <u>relatively</u> low

problems: Relation to physics theory

(a) 'taste violation'

(b) anomaly counting

(3) The Dirac operator having positive and negative eigen values causes a numerical problem (in particular in the chiral limit),

$$\det(\mathcal{D}+m) \geq 0$$

the so called sign problem.

For non-vanishing density (or chemical pot. μ)

$$\mathcal{D} \to \mathcal{D} + i\gamma_0\mu$$

this is most severe: NP-hard

Non-deterministic Polynomial time -hard