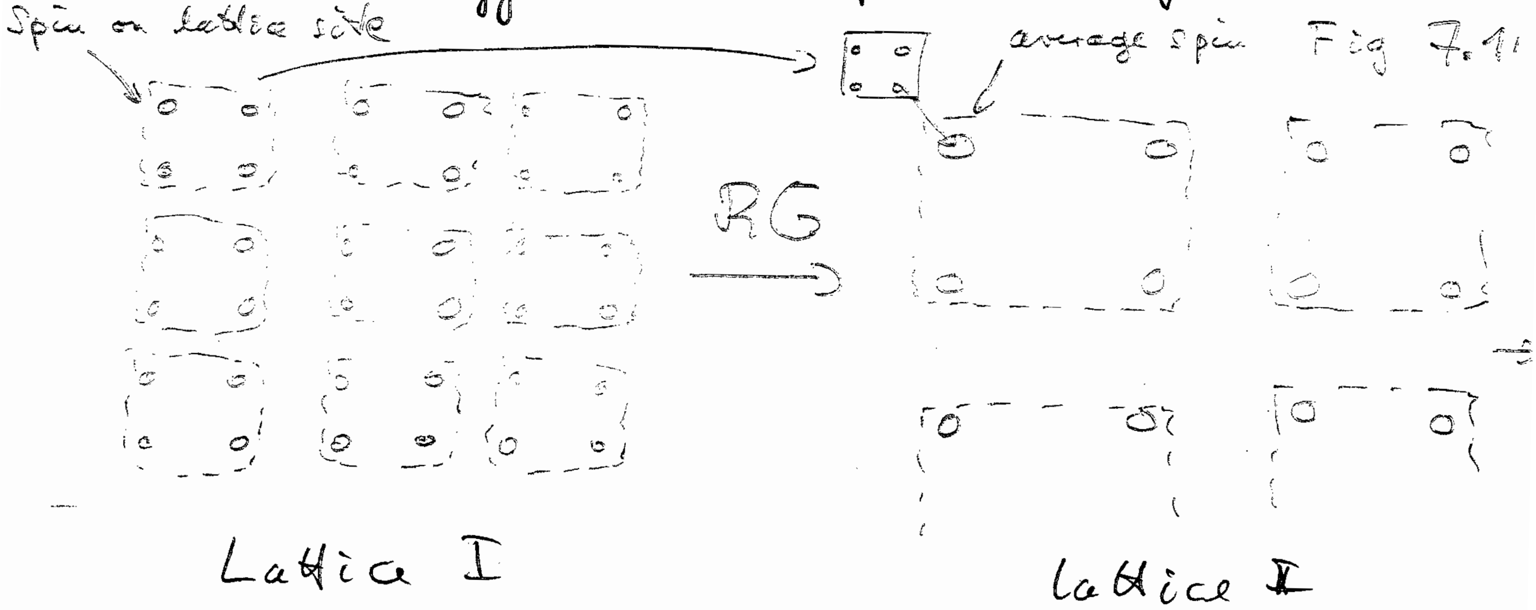
7 Renormalisation group

We have seen in the last chapters, in particular in 6.4, how to use relations, that governed the dependence of correlation functions & observables on physics scales, such as the distance of a $q\bar{q}$-pair, as well as the independence of physics on regularisation scales such as the lattice spacing a.

On the lattice the latter is formalised in the Kadanoff block spin transformations



Also rescaling the lattice parameters (couplings) & fields leads to the same theory, keeping the scales halves (or doubles) the coarse graining scale a. Iterating this procedure, the system 'runs' into a fixed point of the renormalisation group map RG.

7.1 Wilson's renormalisation group

Now we transport this picture to the continuum, with a momentum cut-off $\Lambda \sim 1/a$, and a blocking step going from $\Lambda \to b\Lambda$. On the level of the gen. funct.
($b = 1/2$ in Fig 7.1)
this means (without source terms)

$$Z_\Lambda = \int [D\phi]_\Lambda \, e^{-\overbrace{\int d^dx \left\{ \frac{1}{2}\phi(-\Delta + m^2)\phi + \lambda/4!\, \phi^4 \right\}}^{S[\phi]}} \tag{7.1}$$

$$\text{with } [D\phi]_\Lambda = \prod_{p^2 \leq \Lambda} d\phi(p)$$

or $\phi(p^2 \geq \Lambda^2) = 0$

Now we perform the RG step. To that end we define the field $\hat{\phi}(p)$ with

$$\hat{\phi}(p) = \begin{cases} \phi(p) & b\Lambda \le |p| \le \Lambda \\ 0 & \text{else} \end{cases} \tag{7.2}$$

Now we write the fund. integral measure in eq. (7.1) as

$$[\mathcal{D}\phi]_\Lambda = [\mathcal{D}\phi]_{b\Lambda}\, \mathcal{D}\hat{\phi} \tag{7.3}$$

and hence

$$Z_\Lambda = \int [\mathcal{D}\phi]_{b\Lambda} \int \mathcal{D}\hat{\phi}\; e^{-\int d^dx \left\{ \frac{1}{2}(\phi+\hat{\phi})(-\Delta+m^2)(\phi+\hat{\phi}) + \lambda/4!\,(\phi+\hat{\phi})^4 \right\}}$$

$$= \int [\mathcal{D}\phi]_{b\Lambda}\, e^{-S[\phi]} \cdot \int \mathcal{D}\hat{\phi}\; e^{-\int d^dx \left\{ \frac{1}{2}\hat{\phi}(-\Delta+m^2)\hat{\phi} + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right\}} \tag{7.4}$$

where we have used that (prove with Four. trafo.)

$$\int d^dx\; \hat{\phi}\,\Delta\phi = \int d^dx\; \hat{\phi}\,\phi\, m^2 = 0$$

Integrating formally over $\hat{\phi}$ leads to

$$Z_\Lambda = \int [\mathcal{D}\phi]_{b\Lambda}\, e^{-S_{eff}[\phi]} \tag{7.5}$$

with $e^{-S_{eff}[\phi]} = e^{-S[\phi]} \cdot \int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \left\{ \frac{1}{2}\hat{\phi}(-\Delta + m^2)\hat{\phi} + \dots \right\}}$

Now we assume $\Lambda, b\Lambda$ being bigger than the physics scales, e.g. $m^2/\Lambda^2 \ll 1$. Then we can expand the $\hat{\phi}$-integral in eq. (7.5) about

$$\int d^dx\, \hat{\phi}(-\Delta)\hat{\phi} = \int_{b\Lambda}^{\Lambda} \frac{d^dp}{(2\pi)^d}\, \hat{\phi}(p)\, p^2\, \hat{\phi}(-p) \tag{7.6}$$

$\hookrightarrow b^2\Lambda^2 \le p^2 \le \Lambda^2$

An approximate saddle point is $\hat{\phi} \sim \frac{1}{\Lambda} \to 0$ and hence (see chapter 3*)

$$\int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \left\{ \frac{1}{2}\hat{\phi}(-\Delta + m^2 + \frac{1}{2}\phi^2)\hat{\phi} + \underbrace{\frac{\lambda}{6}\phi^3}_{J(\phi)}\hat{\phi} + O(\hat{\phi}^3) \right\}} \tag{7.7}$$

$$\simeq {\det}'^{\,-1/2}\left(-\Delta + m^2 + \tfrac{\lambda}{2}\phi^2\right) e^{O(1/\Lambda)}$$

$O(1/\Lambda) \leftarrow \frac{1}{2}\int_{x,y} \frac{\lambda}{6}\phi^3 \frac{1}{-\Delta + m^2 + \frac{\lambda}{2}\phi^2} \frac{\lambda}{6}\phi^3$

$\text{"}J\text{"} \longrightarrow \frac{\lambda}{6}\phi^3$

runs over spectral values $b^2\Lambda^2 \le p^2 \le \Lambda^2$

The det in eq. (7.7) can be written in terms of a trace ($\det A = e^{\mathrm{Tr} \ln A}$), to wit

$$\det{}^{-1/2}\left(-\Delta + m^2 + \lambda/_2\, \phi^2\right) = e^{-\frac{1}{2}\mathrm{Tr}' \ln\left(-\Delta + m^2 + \lambda/_2\, \phi^2\right)}$$

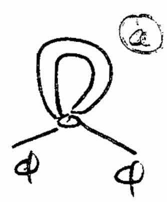
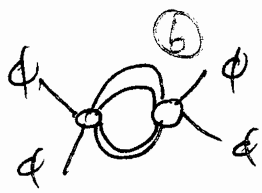
runs over $b^2\Lambda^2 \le p^2 \le \Lambda^2$

(7.8)

and hence

$$S_{eff}[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}' \ln\left(-\Delta + m^2 + \lambda/_2\, \phi^2\right) + O(1/\Lambda) \qquad (7.9)$$

Diagrammatically this yields

$S_{eff}[\phi] = S[\phi]$ +  +  $+ O(\phi^6)$

(7.10)

with $=\!= \; = \left.\frac{1}{p^2 + m^2}\right|_{p^2 \in [b^2\Lambda^2, \Lambda^2]} \simeq \frac{1}{p^2}\left(1 + O\left(m^2/\Lambda^2\right)\right)$

The first two terms are, ⓐ $= \frac{1}{2}\mu^2 \int d^d x\, \phi^2$ with

$$\mu^2 \simeq \frac{\lambda}{4}\int_{b\Lambda}^{\Lambda} \frac{d^d p}{(2\pi)^d}\,\frac{1}{p^2} = \frac{\lambda}{(4\pi)^{d/2}\,\Gamma(d/2)}\,\frac{1 - b^{d-2}}{d-2}\,\Lambda^{d-2}$$

(7.11)

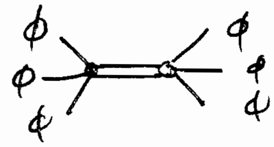
and $+ \frac{1}{4!}\, h \int d^d x\, \phi^4$

$$\simeq -4!\, \frac{1}{2}\, \frac{1}{2} \left(\frac{\lambda}{2}\right)^2 \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{p^2}\right)^2$$

$$= -\frac{3\lambda^2}{(4\pi)^{d/2}\,\Gamma(d/2)} \cdot \frac{(1-b^{d-4})}{d-4}\, \Lambda^{d-4} \qquad (7.12)$$

$$\xrightarrow[d\to 4]{} -\frac{3\lambda^2}{16\pi^2} \ln 1/b$$

From the 'source' term in eq. (7.7) we also get terms with ϕ^6 (and higher),



$$\sim \frac{1}{\Lambda^2} \int_x \phi^6 \qquad (7.13)$$

There are also derivative terms such as

$$\eta \int_x \phi^2 (\partial_\nu \phi)^2 \qquad (7.14)$$

with $\eta \sim \frac{1}{\Lambda^2}$. In summary we have

$$S_{eff}[\phi] = S_{cl}[\phi] + \text{connected diagrams} \qquad (7.15)$$

In the diagrams (coefficients) ⓐ & ⓑ computed in eqs. (7.11), (7.12) we see how the coefficient of the classical action in the generating functional change, if a momentum shell $p^2/\Lambda^2 \in [b^2, 1]$ is integrated out: the UV-cut-off is lowered by the factor b.

From eq. (7.14) we deduce that

$$S_{eff}[\phi] = \int d^d x \left\{ \frac{1}{2} (1 + \overset{\text{only at 2-loop, or } 2^{nd} \text{ it.}}{\Delta Z}) \phi(-\Delta)\phi + \frac{1}{2}\left(m^2 + \underbrace{\Delta m^2}_{\mu^2 \leftarrow \text{eq.}(7.11)}\right)\phi^2 \right.$$

$$\left. + \frac{1}{4!}\left(1 + \underbrace{\Delta\lambda}_{+h \leftarrow \text{1-loop, eq. } (7.12)}\right)\phi^4 + (\overset{\alpha=0}{\alpha + \Delta\alpha})(\partial_\nu \phi)^4 + \frac{\overset{\lambda_6=0}{\lambda_6} + \Delta\lambda_6}{6!}\phi^6 + \dots \right. \qquad (7.16)$$

Now we want to exploit and maximise the similarity of the generating functional as represented in eq. (7.1) and eq. (7.5).

To that end we <u>re-parameterise</u> our theory such that

$$S_{eff}[\phi] = S[\phi'] + \text{higher-order terms}$$

where $\phi(p^2 > b^2\Lambda^2) = 0$ and $\phi'(p'^2 > \Lambda^2) > 0$.

This is achieved by the rescalings

$$p' = p/b \quad , \quad x' = x\,b$$

(see lattice on p. 164)

and

$$\phi'(x') = \left[b^{2-d}(1+\Delta Z)\right]^{1/2}\phi(x) \tag{7.17}$$

$$m'^2 = (m^2 + \Delta m^2)\,\frac{1}{1+\Delta Z}\; b^{\boxed{-2}}$$

momentum dim

ϕ^2 $4-d$

$$\lambda' = (\lambda + \Delta\lambda)\,\frac{1}{(1+\Delta Z)^2}\; b^{\boxed{d-4}}$$

ϕ^4 ∂^4 $-d$

$$\alpha' = (\alpha + \Delta\alpha)\,\frac{1}{(1+\Delta Z)^2}\; b^{\boxed{-d}}$$

$6-2d$

$$\lambda_6' = (\lambda_6 + \Delta\lambda_6)\,\frac{1}{(1+\Delta Z)^3}\; b^{\boxed{2d-6}} \tag{7.18}$$

ϕ^6

with

$$\int d^dx' = b^d \int d^dx \quad , \quad \frac{\partial}{\partial x'} = \frac{1}{b}\,\frac{\partial}{\partial x} \tag{7.19}$$

With the reparameterisations (7.17), (7.18), (7.19) we have

$$S_{eff}[\phi'] = \overbrace{\int d^dx' \left\{ \frac{1}{2}\, \phi'(x')\left(-\Delta' + m'^2\right)\phi' + \frac{\lambda'}{4!}\phi'^4 \right\}}^{S[\phi';\, m', \lambda']}$$

$$+ \int d^dx' \left\{ \alpha' (\partial_\nu' \phi')^4 + \frac{\lambda_6'}{6!}\, \phi'^6 + \dots \right\} \qquad (7.20)$$

The generating functional now reads

$$Z_\Lambda = \int [\mathcal{D}\phi']_\Lambda \, e^{-S_{eff}[\phi']} \qquad (7.21)$$

and hence has the same form as eq. (7.4) with $S \to S_{eff}$. The (Wilsonian) effective action S_{eff} now encodes also the quantum effects of the momentum shell $[b^2\Lambda^2, \Lambda^2]$.

This is not only reflected in the additional terms in the second line in eq. (7.20) but also in the change in the parameters of the theory m, λ, $[\phi]$!

Remark: Assume now that we iterate the above procedure by starting with eq. (7.21) instead of eq. (7.1): The saddle point expansion used in eq. (7.7) still holds (the suppression argument with $1/\Lambda$) and we arrive finally again at $\boxed{\text{eq. (7.21)!}}$ with a Wilson action

$$S_{eff,2} \tag{7.22}$$

where 2 stands for second iteration.

Repeating this we finally have

$$Z = \int [\mathcal{D}\phi]_\Lambda \, e^{-S_{eff_\Lambda}[\phi]} \tag{7.23}$$

where $S_{eff_\Lambda} = S_{eff,\infty}$, the fixed point of the RG-map with $\Lambda \to b\Lambda$.

What we have done is to <u>integrate-out</u> momenta in the shell $[b^2\Lambda^2, \Lambda^2]$ and then we have rescaled the theory such that $b\Lambda \to \Lambda$. Hence we have gone back to the original theory. Then, however, S_{eff_Λ} is <u>not</u> the classical action of the theory, but has to encode the integrated-out quantum fluctuations of all momenta above Λ:

(1) intergrate out $p^2 \in [b^2\Lambda^2, \Lambda^2]$

(2) $p \to p/b$: (1) then entails that we have integrated out $p'^2 \in [\Lambda^2, \frac{1}{b^2}\Lambda^2]$

(3) iterate : $(1)^\infty$ then entails integrating-out $p'^2 \in [\Lambda^2, \infty)$

$$\Rightarrow \boxed{S_{\mathrm{eff}_\Lambda}[\phi] \simeq -\ln \int [D\hat{\phi}]_{p^2 \geq \Lambda^2}\, e^{-S[\hat{\phi}+\phi]} \quad \text{with } \phi(p^2 > \Lambda^2) = 0} \qquad (7.24)$$

What we have gained is that the non-trivial integration in eq. (7.24) can be done in continuous <u>infinitesimal</u> steps in b and we can monitor the <u>flow</u> of couplings, correlation fcts. & S_{eff_Λ}.

In particular we have

$$\boxed{b \frac{d}{db} S_{eff_\Lambda} = 0,} \qquad (7.25)$$

the renormalisation group equation for S_{eff_Λ}, compare with eq. (6.62) on page 151 on the lattice.

Due to the above arguments, the information of (7.25) also tells us about

$$\boxed{\Lambda \frac{d}{d\Lambda} S_{eff_\Lambda},} \qquad (7.26)$$

the physics change when lowering the cut-off.

As one can see from eq. (7.18), the coefficient $\lambda_{n,m}$ of a term (operator) in the effective action $S_{eff,\Lambda}$ with n fields, ϕ^n, and m derivatives, ∂^m, scales like

$$\lambda_{n,m} \xrightarrow{b} b^{\overbrace{\underset{[\phi]\cdot n}{n(d/2-1)} + \underset{[\partial]\cdot m}{m}}^{d_{n,m}} \underset{[d^dx]}{-d}} \lambda_{n,m} \tag{7.27}$$

An operator with

$$d_{n,m} = n(d/2-1) + m \tag{7.28}$$

(a) $d_{n,m} - d > 0$, is called relevant.

(b) $d_{n,m} - d = 0$, is called marginal

(c) $d_{n,m} - d < 0$, is called irrelevant

Relevant ops. grow in the iteration (for cut-off $\to \infty$), irrelevant decay, and marginal ones run logarithmically, see eq. (7.12), p. 169.

Renormalisability:

(1) Irrelevant terms/operators decay like $\Lambda^{d_{n,m}-d}$ for $\Lambda \to \infty$.

(2) Relevant and marginal terms grow with $\Lambda^{d_{n,m}-d}$ for $\Lambda \to \infty$. If the relevant & marginal terms are contained in the classical action, the corrections can be absorbed in redefinitions of the <u>finite number</u> of classical couplings.

We have

$$\boxed{\lim_{\Lambda \to \infty} S_{\mathrm{eff},\Lambda} \to S_{(\mathrm{bare})}[\phi]} \qquad (7.29)$$

Remark: For a full proof it has to be shown that the notions (a), (b), (c) (rel/marg., irrel.) persist at any it. order.
(order pert. theory)