7.2. Fixed points

Fixed points: A fixed point $\underline{S_{eff\,*}}$ of the RG-map / flow implies fixed points in the dimensionless coefficients $\hat{\lambda}_{n,m}$ with

$$\boxed{\hat{\lambda}_{n,m} = \lambda_{n,m} \cdot \Lambda^{d_{n,m}-d}} \qquad (7.29)$$

and hence $\Lambda \frac{d}{d\Lambda} \hat{\lambda}_{\underbrace{n,m}_{i}} = 0$ at the fixed point.

In general we have (see lattice β-fct. (6.63), p. 151)

$$\Lambda \frac{d}{d\Lambda} \hat{\lambda}_i = \hat{\beta}_i(\vec{\hat{\lambda}}) \qquad (7.30)$$

with fixed point $\vec{\hat{\lambda}}_*$,

$$\boxed{\hat{\beta}_i(\vec{\hat{\lambda}}_*) = 0} \qquad (7.31)$$

Example 1: Gaussian fixed point in 4dim ϕ^4:

$$\underbrace{\Lambda \frac{d}{d\Lambda} \hat{\lambda}}_{-\beta_{\hat{\lambda}}} = \mathcal{O}(\overset{\lambda}{\hat{\lambda}^2}), \qquad \overset{-\beta_{\hat{m}^2}}{\Lambda \frac{d}{d\Lambda} \hat{m}^2} = -2\hat{m}^2 + \mathcal{O}(\hat{\lambda}) \qquad (7.32)$$

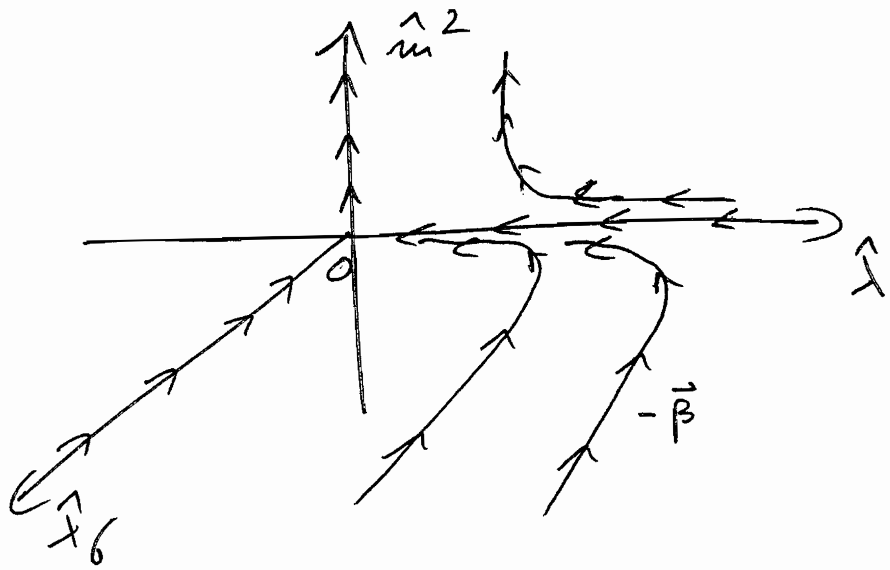
with solution $\boxed{\hat{\lambda}_* = 0 = \lambda_*, \quad \hat{m}^2 = 0}$ (7.33)

For the leading factor for $\hat{\beta}_{\hat{\lambda}}$ the one-loop term $\Delta\lambda = -h = -\frac{3\lambda^2}{16\pi^2} \ln 1/6$ is required, eq. (7.12)

This entails that

$$\boxed{\Lambda \frac{d}{d\Lambda} \hat{\lambda} = \frac{3\hat{\lambda}^2}{16\pi^2}} \qquad (7.34)$$

Now we visualise the situation in a plot of $-\vec{\beta}$ indicating the flow towards the infrared



with eq. (7.32) and $\hat{\beta}_{\hat{\lambda}_6} = \Lambda \frac{d}{d\Lambda} \hat{\lambda}_6 = 2\hat{\lambda}_6$ (7.35)

$+ O(\hat{\lambda}^3, \hat{\lambda}_6 \hat{\lambda}, \hat{\lambda}_8)$

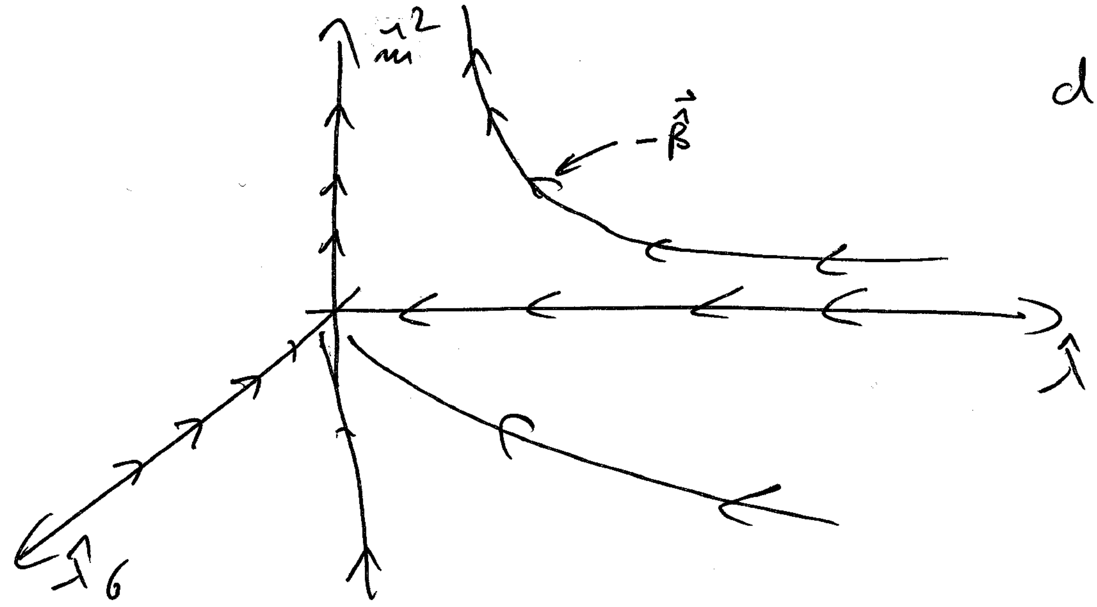
Away from the $\hat{\lambda}$-axis the flow is dominated by the dimension factors in $\hat{\beta}_{\hat{m}^2}$ & $\hat{\beta}_{\hat{\lambda}_6}$.

Example 2: general d

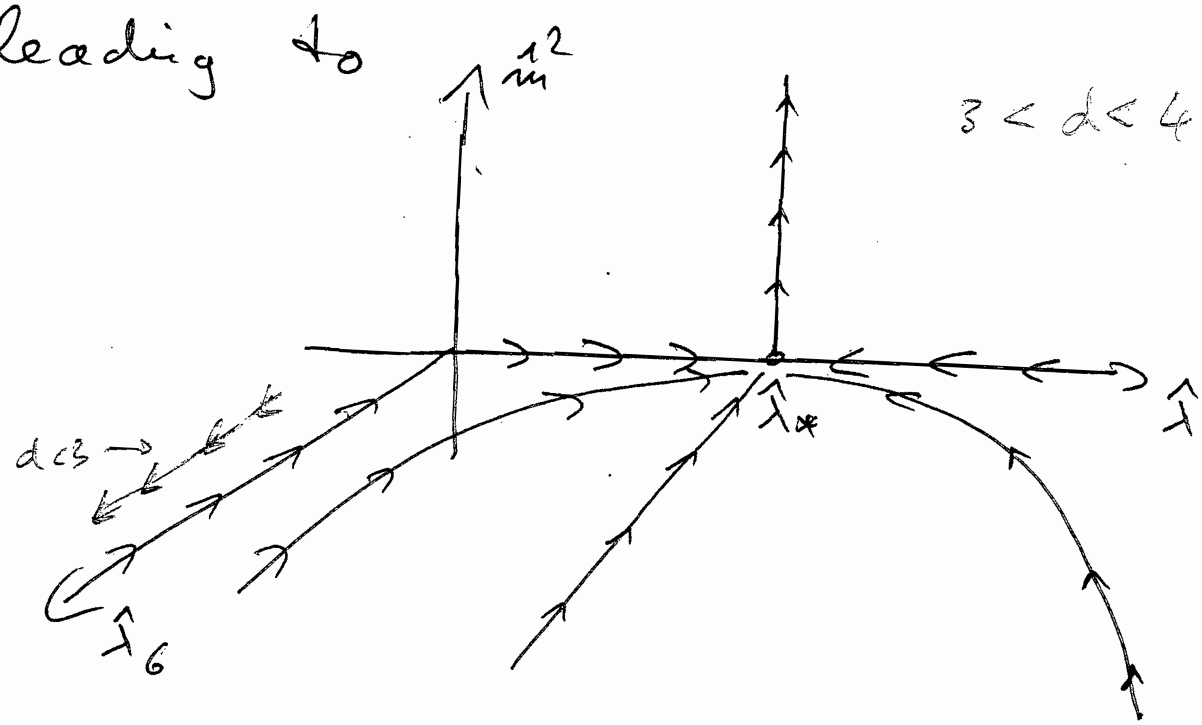
$$\hat\beta_{\hat m^2} = -2\,\hat m^2 + \mathcal{O}(\hat\lambda)$$

$$\hat\beta_{\hat\lambda} = -(4-d)\,\hat\lambda + \frac{3\hat\lambda^2}{16\pi^2} + \mathcal{O}(\hat\lambda_6) \qquad (7.36)$$

$$\hat\beta_{\hat\lambda_6} = -(6-2d)\,\hat\lambda_6 + \mathcal{O}(\hat\lambda^3_j\,\hat\lambda_6\hat\lambda, \hat\lambda_8)$$



For $d<4$ the dimensional part of $\hat\beta_{\hat\lambda}$ is negative and the infrared flow leads to a growing coupling. However, at some coupling $\hat\lambda_*$ the $\hat\lambda^2$ corrections in $\hat\beta_{\hat\lambda}$ in eq. (7.36) will balance the first dimensional term, leading to $\boxed{\hat\beta_{\hat\lambda}(\hat\lambda_*)=0}$ (7.37) leading to



with
$$\boxed{\lambda_* = \frac{16\pi^2}{3}(4-d)} \qquad (7.38)$$

$$\lim_{d\to 4} \lambda_* = 0$$

Remarks:

(1) The $\hat{\beta}$-functions in eq. (7.36) gave us direct access to global scaling features of the ϕ^4-theory.

(2) The stability of fixed points (physics around the fixed points can be accessed in an expansion about $\vec{\lambda}_*$.

Fixed point: $\hat{\beta}_i(\vec{\hat{\lambda}}_*) = 0$

Now we expand about $\vec{\hat{\lambda}}_*$:

$$\hat{\lambda}_i = \hat{\lambda}_{*i} + \delta\hat{\lambda}_i \qquad (7.39)$$

and hence

$$\hat{\beta}_i(\vec{\hat{\lambda}}) = \underbrace{\hat{\beta}_i(\vec{\hat{\lambda}}_*)}_{0} + \hat{B}_{ij}(\vec{\hat{\lambda}}_*)\,\delta\hat{\lambda}_j + \mathcal{O}(\delta\hat{\lambda}_j^2) \qquad (7.40)$$

with $\hat{B}_{ij} = \dfrac{\partial\hat{\beta}_i}{\partial\hat{\lambda}_j}$ Stability matrix

For the flow trajectories around the fixed point $\vec{\hat{\lambda}}_*$ we look for eigen vectors e_i

with $\hat{B}\cdot e_i = \hat{b}_i\, e_i$ (7.41)

and $\delta\vec{\hat{\lambda}} = \sum_i \delta\bar{\lambda}_i\, e_i$ (7.42)

where $\delta\bar{\lambda}_i = \delta\vec{\hat{\lambda}}\cdot e_i$ and $\bar{\lambda}_i = \vec{\hat{\lambda}}\cdot e_i$

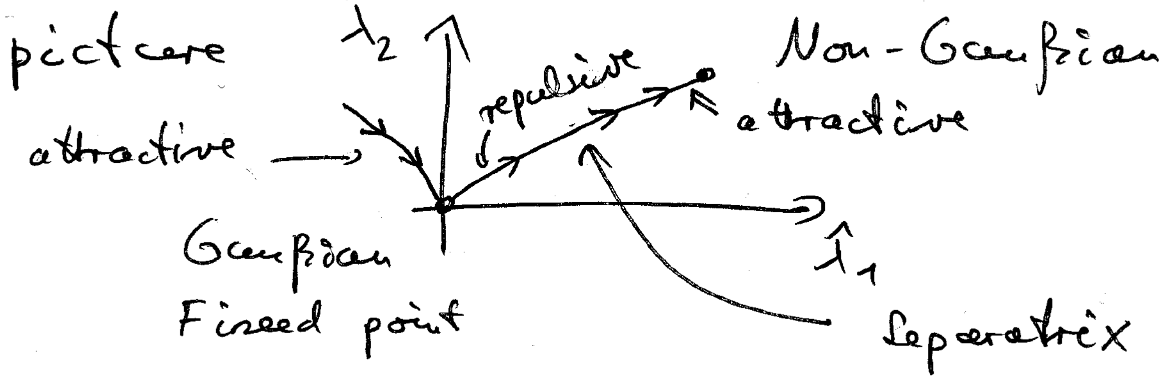
This leads to

$$\Lambda\frac{d}{d\Lambda}\bar{\lambda}_i = \hat{b}_i\,\delta\bar{\lambda}_i \qquad (7.43)$$

with solution

$$\boxed{\bar{\lambda}_j = \bar{\lambda}_{*i} + \bar{\lambda}_{0i} \left(\Lambda/\Lambda_0\right)^{b_i}} \qquad (7.44)$$

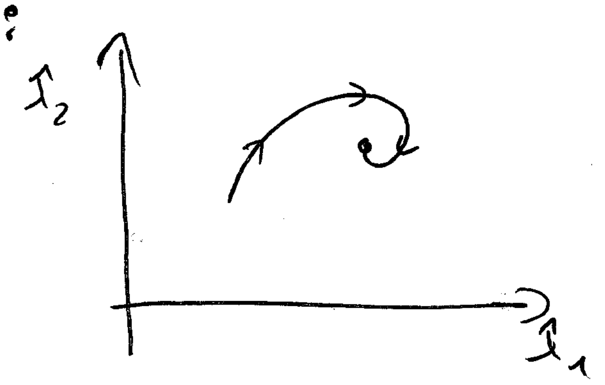
Note that the eigenvalues b_j could be complex. This leads to the general picture



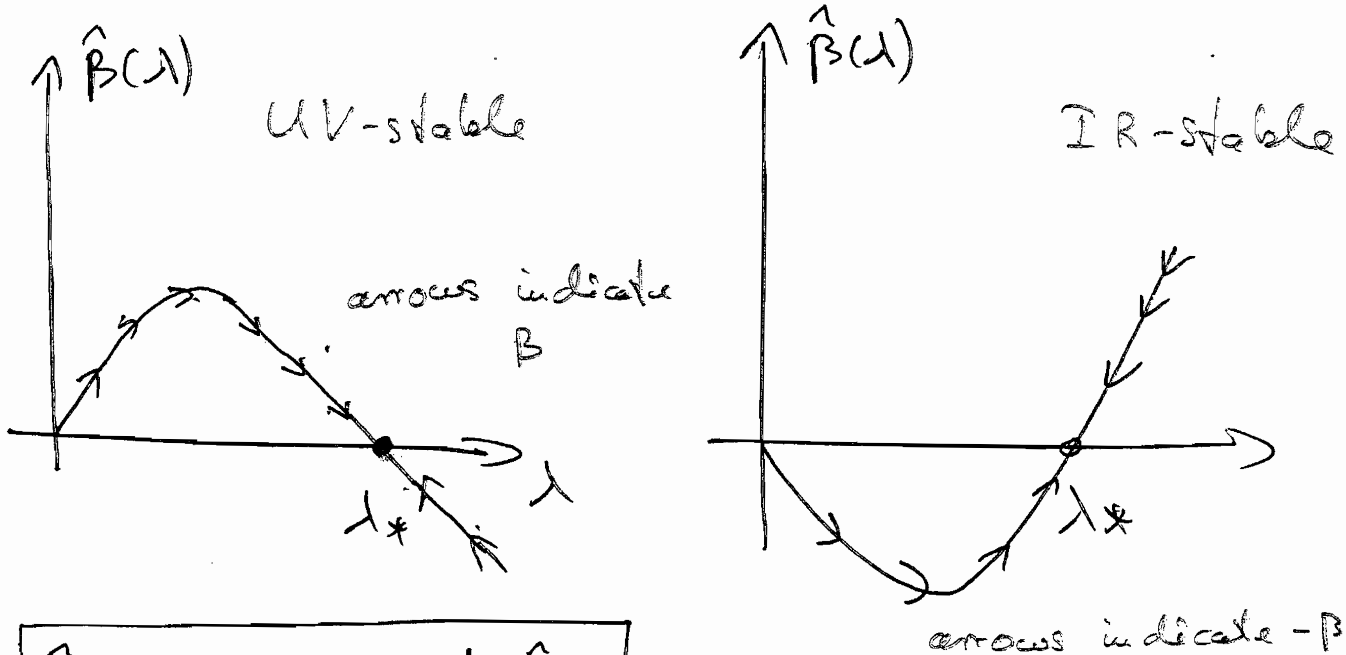
(UV) Attractive: $\hat{b}_i < 0$

(UV) Repulsive: $\hat{b}_j > 0$

$\hat{b}_i$ complex:



The UV/IR-stability is best studied at an even simpler one-dim. example in coupling space. Consider $\hat{\beta}(\lambda)$ with



with $\boxed{\hat{\beta}(\lambda) = \Lambda \frac{d}{d\Lambda} \hat{\lambda}}$

Remark: The Λ-dep. of couplings the momentum dependence of physics:

$S_{eff,\Lambda}[\phi]$ is the effective action including quantum flucs with momenta $p^2 \in [\Lambda^2, \infty]$.

critical exponents