

8.2 Anomalies

We consider a fermionic theory with massless fermions,

$$S[A, \psi, \bar{\psi}] = - \int d^4x \bar{\psi} \not{\partial} \psi \quad (8.27)$$

coupled to a $U(1)$ gauge field. The action in eq. (8.27) is invariant under axial transformations

$$\psi \rightarrow e^{i\alpha \gamma_5} \psi = \psi^\alpha \quad (8.28)$$

leading to the Noether current

$$j_{5\mu} = \bar{\psi} \gamma_\mu \gamma_5 \psi \quad \text{with} \quad \boxed{\partial_\nu j_{5\nu} = 0} \quad (8.29)$$

How does eq. (8.29) look after fermionic quantisation?

To that end, let us consider

$$\int d^4x \langle \partial_\nu j_{5\nu} \rangle \quad (8.30)$$

We compute first in the presence of a

small mass: $\bar{\psi} \not{D} \psi \rightarrow \bar{\psi} (\not{D} + m) \psi$

$$\int d^4x \partial_\nu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = 2m \int d^4x \langle \bar{\psi} \gamma_5 \psi \rangle \quad (8.31)$$

The rhs of eq. (8.31) can be computed as

$$2m \int d^4x \langle \bar{\psi} \gamma_5 \psi \rangle = -2m \int d^4x \text{tr} \gamma_5 \langle \psi \bar{\psi} \rangle$$

$$Z[\eta, \bar{\eta}] = e^{\int \bar{\eta} \frac{1}{\not{D} + m} \eta} \longrightarrow = 2m \int d^4x \text{tr} \gamma_5 \frac{1}{\not{D} + m} \quad (8.32)$$

The integral and Dirac trace can be rewritten

in a sum of eigenvalues of \not{D} :

$$\not{D} |\psi_n\rangle = \lambda_n |\psi_n\rangle$$

$$\Rightarrow \not{D} \gamma_5 |\psi_n\rangle = -\lambda_n \gamma_5 |\psi_n\rangle$$

\uparrow non-zero
 \downarrow EV are paired.

$$(8.33)$$

We conclude

$$2m \int d^4x \text{tr} \gamma_5 \frac{1}{\not{D} + m} = 2m \sum_n \langle \psi_n | \gamma_5 | \psi_n \rangle \frac{1}{\lambda_n + m}$$

$$\uparrow = \frac{2m}{m} \sum_{n_0} \langle \psi_{n_0} | \gamma_5 | \psi_{n_0} \rangle$$

$$\langle \psi_{n_0} | \gamma_5 | \psi_{n_0} \rangle = 0 \quad \text{for } \lambda_{n_0} \neq 0$$

$$(8.34)$$

see eq. (8.33)

where $\mathcal{D} | \psi_0 \rangle = 0$, $| \psi_0 \rangle$ are the zero modes. The ψ_0 can be split into positive and negative chirality modes,

$$\gamma_5 | \psi_0 \rangle = \pm | \psi_0 \rangle, \quad (8.34)$$

as \mathcal{D} commutes with γ_5 on the zero mode space.

We conclude

$$2m \int d^4x \operatorname{tr} \gamma_5 \frac{1}{\mathcal{D} + m} = 2(N_+ - N_-), \quad (8.36)$$

where N_+ , N_- are the number of zero modes with pos. and neg. chirality respectively:

right-handed and left-handed, $\frac{(1 \pm \gamma_5)}{2} | \psi_{0\pm} \rangle = \pm | \psi_{0\pm} \rangle$ (8.37)

This leaves us with the puzzling result that the rhs of eq. (8.32) is non-zero, indeed it is the so-called (analytic) index of \mathcal{D} , whereas the lhs is a total deriv.

Since the fermion is massive, we certainly have

$$\int d^4x \partial_\nu \langle \bar{\psi} \gamma_5 \gamma_\nu \psi \rangle = 0 \quad (8.38)$$

What has gone wrong? We have implicitly used that

$$\int \mathcal{D}\psi^\alpha \mathcal{D}\bar{\psi}^\alpha = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} J^{-1}(x) \quad (8.39)$$

with $J(x) = 1$, as the transformation (8.28) is unitary.

Careful computation:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} = \int \prod_n d a_n \prod_n d \bar{b}_n \quad (8.40)$$

with $\psi(x) = \sum a_n \psi_n(x)$

and $\psi_n(x) = \gamma_5 \psi_n(x)$

$\bar{\psi}(x) = \sum \bar{b}_n \psi_n^\dagger(x)$

$$(8.41)$$

We have

$$\psi^\alpha(x) = \sum a_n e^{i\alpha(x)\gamma_5} \psi_n(x)$$

$$\bar{\psi}^\alpha(x) = \sum \bar{b}_n \psi_n^\dagger(x) e^{i\alpha(x)\gamma_5}$$

$$(8.42)$$

The new coefficient a_n^α is

$$\begin{aligned} a_n^\alpha &= \int d^4x \psi_n^\dagger(x) \psi^\alpha(x) = \int d^4x \sum_m a_m \psi_n^\dagger(x) e^{i\alpha(x)\gamma_5} \psi_m(x) \\ &= a_n + i \sum_m \int d^4x \alpha(x) \psi_n^\dagger(x) \gamma_5 \psi_m(x) \\ &\quad + \mathcal{O}(\alpha(x)^2) \end{aligned} \quad (8.43)$$

We conclude that in leading order (linear) in $\alpha(x)$, the Jacobian J is given by

$$J(\alpha) = e^{2i \sum_n \int d^4x \alpha(x) \psi_n^\dagger(x) \gamma_5 \psi_n(x)} \quad (8.44)$$

It is left to compute

$$\sum_n \psi_n^\dagger(x) \gamma_5 \psi_n(x) \cdot \left[e^{-\varepsilon \delta_n^2} \right] \quad (8.45)$$

We regularise eq. (8.45) with $e^{-\varepsilon \delta^2}$ and get

$$\begin{aligned} &\sum_n \psi_n^\dagger(x) \gamma_5 e^{-\varepsilon \delta_n^2} \psi_n(x) \\ &\stackrel{\langle x | \psi_n \rangle}{=} \int \frac{d^4p}{(2\pi)^4} \text{tr} \gamma_5 e^{-\varepsilon \left((\vec{p} + g\vec{A}_\nu + igA_\nu \partial_\nu)^2 + \frac{i}{2} \sigma_{\nu\rho} F_{\nu\rho} \right)} \\ &= -\frac{1}{8} \int \frac{d^4p}{(2\pi)^4} \text{tr} \gamma_5 \sigma_{\nu\rho} \sigma_{\rho\sigma} F_{\nu\rho} F_{\rho\sigma} e^{-\varepsilon p^2} + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.46)$$

