Problems for Quantum Field Theory 2

11. Sheet

Suggested reading before solving these problems: Chapter 6.1 & 6.2 in the script and/or chapter 3 in *Rothe: Lattice Gauge Theories*.

Problem 1: *Discrete derivatives*

Consider a smooth function $f(x)$. Assume that for some reason you have access to this function only on a discrete set of points, say x, $x \pm a$, $x \pm 2a$, $x \pm 3a, \ldots$. Prove the following relations

$$\begin{aligned}
&\frac{f(x+a)-f(x)}{a} = \frac{f(x)-f(x-a)}{a} = f'(x) + \mathcal{O}(a),\\
&\frac{f(x+a)-f(x-a)}{2a} = f'(x) + \mathcal{O}(a^2),\\
&\frac{f(x+a)+f(x-a)-2f(x)}{a^2} = f''(x) + \mathcal{O}(a^2).
\end{aligned}$$

By using more points, find expressions for $f''(x) + \mathcal{O}(a^3)$ and $f' + \mathcal{O}(a^3)$.

Problem 2: *Scalar field on the lattice*

Consider the action for a free scalar field in Euclidean space

$$S = \int d^4x \frac{1}{2}\phi(x)\left(-\partial_\mu\partial_\mu + M^2\right)\phi(x).$$

A lattice formulation is obtained by making the following substitutions

$$\begin{aligned}
&x \to na \qquad \text{with} \qquad n = (n_1, n_2, n_3, n_4),\\
&\phi(x) \to \phi_n = \phi(na),\\
&\int d^4x \to a^4 \sum_n,\\
&\partial_\mu\partial_\mu\phi(x) \to \frac{1}{a^2}\widehat{\Delta}\phi(na).
\end{aligned}$$

a) Convince yourself that a useful choice for the discrete Laplace operator is

$$\widehat{\Delta}\phi_n = \sum_{\mu=1,2,3,4}\left\{\phi_{n+e_\mu} + \phi_{n-e_\mu} - 2\phi_n\right\},$$

and that the action becomes with this choice

$$S = -\frac{a^2}{2}\sum_{n,\mu}\phi_n\phi_{n+e_\mu} + \frac{a^2}{2}(8 + a^2M^2)\sum_n \phi_n\phi_n.$$

What is here the summation range of μ?

b) This action can be written in the form

$$S = \frac{a^4}{2} \sum_{n,m} \phi_n K_{nm} \phi_m.$$

Derive an explicit expression for the matrix K_{nm}.

c) Use the following representation of the Kronecker delta

$$\frac{1}{a^4}\delta_{nm} = \int_{-\pi/a}^{\pi/a} \frac{d^4q}{(2\pi)^4} e^{iq\cdot(n-m)a}$$

to obtain the inverse propagator in momentum space which is defined by

$$K_{nm} = \int_{-\pi/a}^{\pi/a} \frac{d^4q}{(2\pi)^4} K(q) e^{iq\cdot(n-m)a}.$$

PROBLEM 3: *Gauge invariance*

In the lecture we have introduced the link variables $U_\mu(n) \in SU(N)$ with the transformation properties

$$U_\mu(n) \xrightarrow{G} G(n) U_\mu(n) G^\dagger(n+\hat{\mu}).$$

a) Show that

$$\hat{\phi}_n^\dagger U_\mu(n) \hat{\phi}_{n+\hat{\mu}}$$

is gauge invariant with the complex scalar field $\hat{\phi}$ having the transformation properties

$$\hat{\phi}_n \xrightarrow{G} G(n)\hat{\phi}_n, \quad \hat{\phi}_n^\dagger \xrightarrow{G} \hat{\phi}_n^\dagger G_n^\dagger.$$

b) Follow the steps is the lecture and show that the scalar action

$$S[\hat{\phi}, U] = -\sum_{n,\mu>0} \left(\hat{\phi}_n^\dagger U_\mu(n) \hat{\phi}_{n+\hat{\mu}} + \phi_n^\dagger U_\mu^\dagger(n-\hat{\mu}) \hat{\phi}_{n-\hat{\mu}} + 2\hat{\phi}_n^\dagger \hat{\phi}_n \right) + \hat{m}^2 \sum_n \hat{\phi}_n^\dagger \hat{\phi}_n \tag{1}$$

is gauge invariant. Here $\hat{m}^2 = a^2 m^2$.

c) Determine the continuum limit of (1).