

PROBLEMS FOR QUANTUM FIELD THEORY 2
12. Sheet

Suggested reading before solving these problems: Chapter 6.2-6.4 in the script and/or chapter 6, 7, 11.1 in *Rothe: Lattice Gauge Theories*.

PROBLEM 1: *Strong coupling expansion in Lattice QCD*

Consider $SU(3)$ Yang-Mills theory on the lattice. The action is given by

$$S = -\beta \sum_P S_P$$

where $\beta = \frac{6}{g_0^2}$ and

$$S_P = \frac{1}{6} \text{Tr}(U_P + U_P^\dagger)$$

is the contribution associated with a plaquette P . The sum goes over all distinct plaquettes. U_P is the plaquette variable and can be expressed in terms of the link variables around the plaquette P as

$$U_P = U_{\mu\nu}(n) = U_\mu(n)U_\nu(n + \hat{\mu})U_\mu^\dagger(n + \hat{\nu})U_\nu^\dagger(n).$$

The Wilson loop with spatial extend \hat{R} and temporal extend \hat{T} (in units of the lattice spacing a) can be expressed in terms of the link variables by the path ordered product

$$W_C[U] = \text{Tr} \prod U_l$$

of link variables around the loop.

Consider now the expectation value

$$\langle W_C[U] \rangle = \frac{\int DU W_C[U] e^{-S[U]}}{\int DU e^{-S[U]}}.$$

Expand the exponential in powers of the coupling β

$$e^{-S[U]} = e^{\beta \sum_P S_P} = \prod_P \left[\sum_n \frac{\beta^n}{n!} (S_P)^n \right]$$

and use the following integration rules for the integrals over the link variables $U \in SU(3)$

$$\begin{aligned} \int dU U^{ab} &= 0, \\ \int dU U^{ab} U^{cd} &= 0, \\ \int dU U^{ab} (U^\dagger)^{cd} &= \frac{1}{3} \delta_{ad} \delta_{bc}, \\ \int dU 1 &= 1, \end{aligned}$$

to show

$$\langle W_C[U] \rangle \sim \left(\frac{\beta}{c} \right)^{\hat{R}\hat{T}}$$

with some coefficient c .

The Wilson loop behaves for large \hat{T} as

$$\langle W_C[U] \rangle \xrightarrow{\hat{T} \rightarrow \infty} F(\hat{R}) e^{-\hat{V}(\hat{R})\hat{T}}$$

so that you can deduce that the potential is linear in \hat{R} :

$$\hat{V}(\hat{R}) = \hat{\sigma}\hat{R},$$

with $\hat{\sigma} = -\ln(\beta/c)$ the string tension.