Problems for Quantum Field Theory 2

2. Sheet

Suggested reading before solving these problems: Chapter 1 in the script and/or chapter 9.1-9.3 in *Peskin & Schroeder*.

Problem 1: *Quantum statistical mechanics*

a) Consider the quantum statistical partition function of the canonical ensemble

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \text{where} \quad \beta = \frac{1}{T},$$

where H is the Hamiltonian. (Units are choosen such that $k_B = 1$.) Use the same strategy that led to the path integral formula for matrix elements of e^{-iHt} in terms of the Lagrangian to derive a similar formula for Z. Show that one has to integrate over functions that are periodic in the "time argument" τ with range from 0 to $1/T$. Note that an Euclidean version of the action

$$S_E = \int_0^{1/T} d\tau L_E$$

appears in the weight.

b) Consider now a one-dimensional harmonic oscillator in an electric field. The Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 - eEx.$$

Show that the Euclidean action appearing in the path integral formula reads

$$S_E = \int_0^{1/T} d\tau \left\{ \frac{1}{2}m\, \dot{x}(\tau)^2 + \frac{1}{2}m\omega^2 x(\tau)^2 - eE\, x(\tau) \right\}.$$

c) Use a Fourier decomposition

$$x(\tau) = T \sum_{n=-\infty}^{\infty} x_n e^{i\omega_n \tau}, \qquad \text{with} \quad \omega_n = 2\pi T n,$$

$$x_n = \int_0^{1/T} d\tau e^{-i\omega_n \tau} x(\tau),$$

to rewrite S_E in terms of the fields x_n.

d) By performing now the path integral show that

$$Z = c\, e^{\frac{(eE)^2}{2mT\omega^2}},$$

where c is independent of the electric field E.

e) Derive an expression for the susceptibility

$$\chi = \frac{\partial \langle e\, x \rangle}{\partial E},$$

where the brackets denote averaging with respect to the thermal distribution.

f) Generalize now the construction of part a) to field theory. Derive an expression for the quantum statistical partition function of a scalar field in terms of a functional integral. Show for a free theory that the value of this integral is proportional to the formal expression

$$\left[\det(-\partial^2 + m^2)\right]^{-1/2},$$

where the operator acts on functions in Euclidean space that are periodic in the time direction with periodicity β.