Problems for Quantum Field Theory 2

3. Sheet

Suggested reading before solving these problems: Chapter 1, 2.1 in the script and/or chapter 9.5, 11.3 –11.5 in *Peskin & Schroeder*.

Problem 1: *Effective potential*

In the lecture you derived an expression for the effective action in one-loop approximation

$$\Gamma_{\text{1-loop}}[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln S^{(2)}[\phi].$$

Consider ϕ^4 theory in d Euclidean dimensions

$$S[\phi] = \int d^dx \left\{\frac{1}{2}\phi(-\partial_\mu\partial_\mu)\phi + V_0(\phi)\right\}, \tag{1}$$

with $V_0(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda_0\phi^4$. Write the effective action in terms of an derivative expansion

$$\begin{aligned}\Gamma_{\text{1-loop}}[\phi] = \quad & \int d^dx \Big\{ \quad \frac{1}{2}\phi(-Z_1\partial_\mu\partial_\mu + Z_2\partial_\mu\partial_\mu\partial_\nu\partial_\nu + \dots)\phi \\ & \qquad + V_{\text{eff}}(\phi)\Big\} + \text{const.}\end{aligned} \tag{2}$$

The first line vanishes for constant fields

$$\Gamma_{\text{1-loop}}[\phi]\Big|_{\phi=\text{const}} = \int d^dx\; V_{\text{eff}}(\phi) + \text{const.}$$

The functional trace operation Tr is best evaluated in momentum space

$$\phi(x) = \int \frac{d^dp}{(2\pi)^d}\;\phi(p)\;e^{ipx}. \tag{3}$$

a) Show that the effective potential is given by

$$V_{\text{eff}}(\phi) = V_0(\phi) + \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\ln\left(p^2 + m_0^2 + \frac{\lambda_0}{2}\phi^2\right) + \text{const.} \tag{4}$$

b) It is often useful to expand the effective potential in a Taylor series. A discrete symmetry $\phi \to -\phi$ implies that only even terms appear

$$V_{\text{eff}}(\phi) = V_{\text{eff}}(0) + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + \dots. \tag{5}$$

Derive formal expressions in terms of integrals for m^2 and λ.

c) The effective action $\Gamma[\phi]$ can be used to generate the amputated n-point functions. Calculate the Feynman diagrams up to 1-loop order for the two- and four-point functions at vanishing external momenta. Show that one obtains indeed the same expressions as derived in part b).

Problem 1: *Grassmann variables: Change of variables*

Consider a set of Grassmann generators $c_1, \ldots, c_n$ and a non-singular matrix A. For $c'_i = A_{ij} c_j$ prove

$$c'_1 \ldots c'_n = (\det A) c_1 \ldots c_n.$$

Show that this implies

$$\int dc_1 \ldots dc_n = \int dc'_1 \ldots dc'_n \, J, \qquad \text{with} \qquad J = \det \frac{\partial c'_i}{\partial c_j}.$$