Problems for Quantum Field Theory 2

4. Sheet

Suggested reading before solving these problems: Chapter 2.2-3.1 in the script and/or chapter 11.3 –11.5 in *Peskin & Schroeder*.

Problem 1: *Effective action*

Consider a scalar field theory in d Euclidean dimensions. The partition function is given by

$$e^{W[J]} = Z[J] = \int d\varphi \, e^{-S[\varphi]+\int d^dx\, J(x)\varphi(x)}.$$

The effective action $\Gamma[\phi]$ is defined as the Legendre transform of the Schwinger functional $W[J]$.

$$\Gamma[\phi] = \sup_J \left(\int d^dx J(x)\phi(x) - W[J] \right).$$

a) Show that this definition implies

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)},$$

$$J(x) = \frac{\delta \Gamma[\phi]}{\delta \phi(x)}$$

and

$$\int d^dz \Gamma^{(2)}(x,z) W^{(2)}(z,y) = \delta^{(d)}(x-y).$$

In a symbolic notation the last line reads

$$\Gamma^{(2)} = \left(W^{(2)}\right)^{-1}.$$

b) Show that the effective action can be written as an implicit functional integral in the presence of a "background field" ϕ

$$e^{-\Gamma[\phi]} = \int D\varphi \, e^{-S[\phi+\varphi]+\int d^dx\, \frac{\delta\Gamma}{\delta\phi}(x)\varphi(x)}. \tag{1}$$

c) A perturbative expansion treats the fluctuations φ about the background ϕ in a saddle point approximation. In the lowest order (tree approximation) one has (up to an additive constant)

$$\Gamma[\phi] = S[\phi].$$

At one-loop order one expands

$$\begin{aligned} S[\phi+\varphi] &= S[\phi] + \int d^dx \, \frac{\delta S[\phi]}{\delta\phi(x)}\varphi(x) \\ &\quad + \frac{1}{2}\int d^dx d^dy S^{(2)}[\phi](x,y)\, \varphi(x)\varphi(y) + \ldots . \end{aligned}$$

Show that this leads to the one-loop expression

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln S^{(2)}[\phi] + \ldots .$$

d) Often one is interested in the expansion of $\Gamma[\phi]$ for small values of the fields ϕ. One can then expand $S^{(2)}$

$$\begin{aligned} S^{(2)}[\phi] &= S^{(2)}[0] + \int d^dx\, S^{(3)}[0](x)\, \phi(x) \\ &\quad + \frac{1}{2}\int d^dx d^dy\, S^{(4)}[0](x,y)\, \phi(x)\phi(y) + \ldots . \end{aligned}$$

By expanding also the logarithm in Eq. (1) show that the effective action has the expansion

$$\Gamma[\phi] = S[\phi] + \Delta\Gamma^{(0)} + \int d^dx\, \Delta\Gamma^{(1)}(x)\phi(x) + \frac{1}{2}\int d^dx d^dy\, \Delta\Gamma^{(2)}(x,y)\phi(x)\phi(y) + \ldots ,$$

with

$$\Delta\Gamma^{(0)} = \frac{1}{2}\mathrm{Tr}\left\{\ln S^{(2)}[0]\right\},$$

$$\Delta\Gamma^{(1)}(x) = \frac{1}{2}\mathrm{Tr}\left\{(S^{(2)}[0])^{-1}S^{(3)}[0](x)\right\}$$

and

$$\Delta\Gamma^{(2)}(x,y) = \frac{1}{2}\mathrm{Tr}\left\{(S^{(2)}[0])^{-1}S^{(4)}[0](x,y) - (S^{(2)}[0])^{-1}S^{(3)}[0](x)(S^{(2)}[0])^{-1}S^{(3)}[0](y)\right\}.$$

e) Can you interpret these expressions in terms of Feynman diagrams?

PROBLEM 2: *Vacuum polarization of Yukawa theory*
Consider Yukawa theory in d Euclidean dimensions

$$S[\psi,\bar\psi,\varphi] = \int d^dx \left\{-\bar\psi(\gamma^\mu\partial_\mu + m)\psi - h\bar\psi\psi\varphi + \frac{1}{2}\varphi(-\partial_\mu\partial_\mu + m_\varphi^2)\varphi\right\}.$$

Following the steps performed in the lecture, show that the boson two-point function has a contribution at one-loop level

$$\langle\varphi(-q)\varphi(p)\rangle_{\text{connected}} = (2\pi)^d\delta^{(d)}(p-q)G_\varphi(p)\Pi(p)G_\varphi(p),$$

with

$$\Pi(p) = -h^2\int\frac{d^dq}{(2\pi)^d}\frac{\mathrm{tr}\left[(-i\gamma^\mu q_\mu + m)(-i\gamma^\nu(p_\nu+q_\nu)+m)\right]}{(q^2+m^2)((p+q)^2+m^2)}.$$

Use $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ for showing that

$$\Pi(p) = -h^2\int\frac{d^dp}{(2\pi)^d}\frac{-d\, q\cdot(q+p) + d\, m^2}{(q^2+m^2)((q+p)^2+m^2)}.$$