Problems for Quantum Field Theory 2

6. Sheet

Suggested reading before solving these problems: Chapter 4.1-4.2 in the script and/or chapter 15 in *Peskin & Schroeder*.

Problem 1: *Representations of Lie algebras*

An element of a Lie group that is close to the identity can be written as

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(a^2).$$

The hermitian operators T^a are the generators of the Lie algebra. They have the commutation relation

$$[T^a, T^b] = if^{abc}T^c, \tag{1}$$

with f^{abc} the structure constants. The vector space spanned by the generators with the additional *Lie bracket* structure in Eq. (1) is called *Lie algebra*.

a) Prove the identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0,$$

and that this implies

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0.$$

b) What is a representation t^a of a Lie algebra and what means irreducible?

c) Assume that the generators in some representation r are normalized according to

$$\mathrm{tr}\{t_r^a t_r^b\} = C(r)\delta^{ab}.$$

Show that this yields the following representation of the structure constants

$$f^{abc} = -\frac{i}{C(r)}\mathrm{tr}\left\{[t_r^a, t_r^b]t_r^c\right\},$$

and that f^{abc} is totally antisymmetric.

d) Consider now SU(N) with generators t_r^a. The fundamental representation is given by

$$\phi \to (1 + i\alpha^a t_r^a)\phi.$$

Show that the matrices $t_{\bar{r}}^a = -(t_r^a)^*$ also lead to a representation (the *conjugate* representation).

e) Similarly, show that the matrices $(t_G^b)_{ac} = if^{abc}$ define a representation (the *adjoint* representation).

PROBLEM 2: *Field strength tensor*

For a non-abelian gauge theory the covariant derivative is given by $D_\mu = \partial_\mu + igA_\mu^a t^a$. The field strength tensor can be defined by

$$[D_\mu, D_\nu] = igF_{\mu\nu}^a t^a.$$

a) Derive the more explicit form of the field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c.$$

b) How do the the field A_μ^a and the field strength tensor $F_{\mu\nu}^a$ transform under infinitesimal and finite local gauge transformations?

c) Show that $F_{\mu\nu}^a F^{a\,\mu\nu}$ is invariant.