Problems for Quantum Field Theory 2

1. Tutorial

Problem 1: *One dimensional Gaussian integrals*

a) Prove the following identity for $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}.$$

b) Show the generalization

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}e^{\frac{b^2}{2a}}.$$

c) Consider now Gaussian integrals over complex numbers $z = \frac{1}{\sqrt{2}}(x+iy)$ with $\int dzdz^* = \int dxdy$. Show for $w \in \mathbb{C}$, $\mathrm{Re}(w) > 0$

$$\int dzdz^*\, e^{-z^*wz} = \frac{\pi}{w},$$

as well as the generalization for arbitrary $u, v \in \mathbb{C}$

$$\int dzdz^*\, e^{-z^*wz+u^*z+z^*v} = \frac{\pi}{w}e^{u^*w^{-1}v}.$$

Problem 2: *Gaussian integrals in more than one dimension*

a) Proof for a real, symmetric and positive definite N-dimensional matrix $\mathbf{A}$ and $\mathbf{x} \in \mathbb{R}^N$ a real N-component vector

$$\int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x^T A x}} = (2\pi)^{N/2}\frac{1}{\sqrt{\det\mathbf{A}}}.$$

(Tip: Use a change of integration variables to diagonalize $\mathbf{A}$.)

b) Proof the generalization

$$\int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x^T A x}+\mathbf{j^T x}} = (2\pi)^{N/2}\frac{1}{\sqrt{\det\mathbf{A}}}\, e^{\frac{1}{2}\mathbf{j^T A^{-1} j}}. \quad (1)$$

c) Show for hermitian and positive $\mathbf{A} = \mathbf{A}^\dagger$

$$\int d\mathbf{z}d\mathbf{z}^\dagger\, e^{-\mathbf{z}^\dagger\mathbf{A}\mathbf{z}+\mathbf{u}^\dagger\mathbf{z}+\mathbf{z}^\dagger\mathbf{v}} = \frac{\pi^N}{\det\mathbf{A}}\, e^{\mathbf{u}^\dagger\mathbf{A}^{-1}\mathbf{v}}.$$

Problem 3: *Expectation values and Wick's theorem*

a) Consider expectation values with respect to the Gaussian weight

$$\langle \dots \rangle = \frac{\int d\mathbf{x}\, (\dots)\, e^{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}}{\int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}}.$$

Using Eq. (1) show

$$\begin{aligned} \langle x_m \rangle &= 0, \\ \langle x_m x_n \rangle &= A^{-1}_{mn}, \\ \langle x_m x_n x_p \rangle &= 0, \\ \langle x_m x_n x_p x_q \rangle &= A^{-1}_{mn} A^{-1}_{pq} + A^{-1}_{mp} A^{-1}_{nq} + A^{-1}_{mq} A^{-1}_{np}, \end{aligned}$$

and more general

$$\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle = \begin{cases} 0 & n \text{ odd} \\ \text{all full contractions} & n \text{ even.} \end{cases}$$

This is a version of Wick's theorem for real bosonic fields.

b) Derive the corresponding relations for complex Gaussian integrals.