Problems for Quantum Field Theory 2

9. Tutorial

Problem 1: *Casimir operator*

Consider a Lie algebra with commutation relation

$$[T^a, T^b] = if^{abc}T^c.$$

Show that the operator $T^2 = T^a T^a$ commutes with all generators, i.e.

$$[T^b, T^a T^a] = 0.$$

This implies that T^2 takes a constant value for each irreducible representation t_r^a and the matrix representation is proportional to the unit matrix

$$t_r^a t_r^a = C_2(r) \cdot 1.$$

Use the normalization

$$\mathrm{tr}[t_r^a t_r^b] = C(r)\delta^{ab}$$

to show the identity

$$d(r)C_2(r) = d(G)C(r),$$

where $d(r)$ is the dimension of the (matrix) representation r and $d(G)$ is the dimension of the adjoint representation.