1 Functional integral approach

1.1 Path integral in quantum mechanics

We aim at correlation fcts. & time evolution of QM systems: (1 dim)

$$\hat{H} = \hat{p}^2/2m + V(\hat{q}) \tag{1.1}$$

with $\hbar = c = 1$

$$[\hat{q}, \hat{p}] = i \qquad \text{Heisenberg alg.} \tag{1.2}$$

Hilbert space $\mathcal{H}$ related to (1.2):

Space of square-integrable fcts.:

$$\Psi : q \to \Psi(q) \tag{1.3}$$

with

$$\int_{\mathbb{R}} dq\, |\Psi|^2(q) < \infty$$

Remark: The above is only a possible rep. of the Heisenberg alg. (1.1) and the Hilbert space.

Representation of ops.:

$$\hat{q}: \quad \hat{q}\,\psi(q) = q \cdot \psi(q)$$
$$\hat{p}: \quad \hat{p}\,\psi(q) = -i\,\frac{\partial \psi}{\partial q}(q) \qquad (1.4)$$
$$\hat{p} = -i\,\frac{\partial}{\partial q}$$

Eigenstates:
$$\hat{p}\,|p\rangle = p\,|p\rangle \qquad (1.5)$$
$$\hat{q}\,|q\rangle = q\,|q\rangle$$

with continuous spectrum $q, p \in \mathbb{R}$.

Normalisation: $\langle q|q'\rangle = 0$ for $q \neq q'$

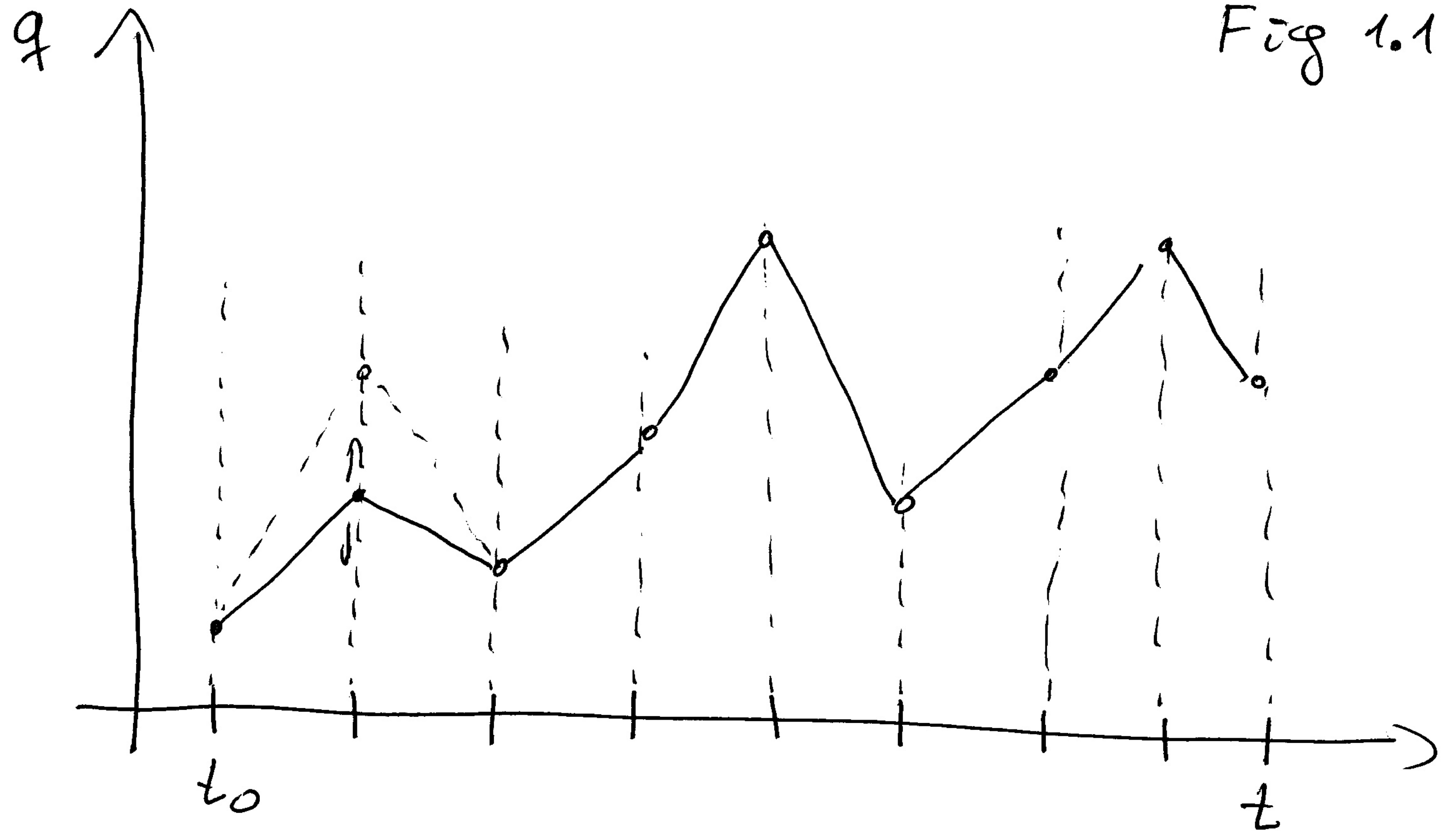
$\int_{\mathbb{R}} dq'\,|q'\rangle\langle q'| = 1 \quad \Rightarrow \quad \int_{\mathbb{R}} dq'\,\langle q|q'\rangle\langle q'| = \langle q|$

$$\Rightarrow \langle q|q'\rangle = \delta(q - q') \qquad (1.6)$$

We also define
$$\langle p|p'\rangle = 2\pi\,\delta(p - p') \qquad (1.7)$$

Hence
$$\langle p|q\rangle = e^{-ipq} \qquad (1.8)$$

Fig 1.1



$$q_i = q(t_i)$$

Correlation fcts. : transition amplitude from initial state $|q_{in}\rangle$ at t_0 to final state $|q_f\rangle$ at $t_f = t$

see QFT I, chapter 3

time evolution op.:

$$\boxed{i\,\partial_t\, \mathcal{U}(t,t_0) = \hat{H}(t)\,\mathcal{U}(t,t_0)} \qquad (1.9)$$

$\Rightarrow$ transition amplitude: $\Delta t = \dfrac{t-t_0}{n}$

$$\langle q_f | \mathcal{U}(t,t_0) | q_{in}\rangle$$

$$= \langle q_f | \mathcal{U}(t, t-\Delta t) \cdots \mathcal{U}(t-(n-1)\Delta t, t_0) | q_{in}\rangle$$

$$= \left[\prod_{i=1}^{n-1} \int dq_i\right] \langle q_f | \mathcal{U}(t,t-\Delta t) | q_{n-1}\rangle\langle q_{n-1} | \mathcal{U}(t-\Delta t, t-2\Delta t) | q_{n-2}\rangle$$

$$\cdots \langle q_1 | \mathcal{U}(t-(n-1)\Delta t, t_0) | q_{in}\rangle \qquad (1.10)$$

where we have used

$$\mathcal{U}(t,t_1)\,\mathcal{U}(t_1,t_2) = \mathcal{U}(t,t_2)$$

for $t \geq t_1 \geq t_2$ (1.11)

For $\Delta t \to 0$: $\quad t_j = t_0 + j\,\Delta t \quad , t = t_n$

$$\mathcal{U}(t_j, t_j - \Delta t) = \underbrace{1 - i\hat{H}\Delta t}_{\sim e^{-i\hat{H}\Delta t}} + O(\Delta t^2) \qquad (1.12)$$

satisfies (1.9).

Hence

$$\langle q_j | \mathcal{U}(t_j, t_{j-1}) | q_{j-1} \rangle$$

$$\simeq \langle q_j | (1 - i\hat{H}\Delta t) | q_{j-1} \rangle$$

$$= \int \frac{dp}{2\pi} \langle q_j | p \rangle \left(\langle p | q_{j-1} \rangle - i \langle p | \hat{H} | q_{j-1} \rangle \Delta t \right)$$

$$= \int \frac{dp}{(2\pi)} e^{+ipq_j} \left(e^{-ip\,q_{j-1}} - i \langle p | \hat{H} | q_{j-1} \rangle \Delta t \right) \qquad (1.13)$$

We use that: (eq. (1.1))

$$\langle p | \hat{H} | q \rangle = \langle p | \hat{p}^2/2m + V(\hat{q}) | q \rangle$$

$$= \left(p^2/2m + V(q) \right) \langle p | q \rangle$$

$$= \left(p^2/2m + V(q) \right) e^{-ipq} \qquad (1.14)$$

Remark: in eq. (1.14) we have used that $\hat{H} = \hat{H}_1(\hat{p}) + \hat{H}_2(\hat{q})$. For general $\hat{H}$ $\langle p | \hat{H} | q \rangle = H(p,q) \langle p | q \rangle$ does <u>not</u> hold.

We conclude

$$\langle q_j | \mathcal{U}(t_j, t_{j-1}) | q_{j-1} \rangle \qquad V(q_{j-1}) = V(q_j) + O(\Delta t)$$

$$\simeq \int \frac{dp}{(2\pi)} e^{ip(q_j - q_{j-1})} \left[1 - i \left(p^2/2m + V(q_j) \right) \Delta t \right]$$

$$\simeq \int \frac{dp}{(2\pi)} e^{i\left\{ p(q_j - q_{j-1}) - \left(p^2/2m + V(q_j) \right) \Delta t \right\}} \tag{1.15}$$

$$q_j - q_{j-1} = \underbrace{\frac{q_j - q_{j-1}}{\Delta t}}_{\text{discrete derivative } \dot{q}_j} \cdot \Delta t$$

It follows:

$$\langle q_f | \mathcal{U}(t, t_0) | q_{in} \rangle \simeq \int \left[\prod_j \frac{dq_j \, dp_j}{(2\pi)^n} \right] \cdot e^{i \Delta t \sum_j \left(\dot{q}_j p_j - H(p_j, q_j) \right)} \tag{1.16}$$

with $H(p, q) = p^2/2m + V(q)$. Finally, $\Delta t \to 0$:

$$\boxed{\langle q_f | \mathcal{U}(t, t_0) | q_{in} \rangle \simeq \int \mathcal{D}q \, \mathcal{D}p \, e^{i \int_{t_0}^{t} dt' \left(\dot{q}(t') p(t') - H(p, q) \right)}} \tag{1.17}$$

with
$$\mathcal{D}q = \prod_j dq_j = \prod_j dq(t_j)\Big|_{\substack{q(t_0)=q_{in}\\ q(t)=q_f}}$$
$$\mathcal{D}p = \prod_j dp_j = \prod_i dp(t_j) \qquad (1.18)$$

Now we use that

$$\int_{\mathbb{R}} \frac{dp}{(2\pi)}\, e^{i(p\dot{q} - p^2/2m)} = \int \frac{dp}{(2\pi)} e^{-i(p-\dot{q}\cdot m)^2/2m} \cdot e^{i\dot{q}^2 m/2}$$
$$= \sqrt{\frac{m}{2\pi}} \cdot e^{i m \dot{q}^2/2} \qquad (1.19)$$

and arrive at

$$\boxed{\langle q_f | \mathcal{U}(t,t_0) | q_{in}\rangle = \frac{1}{N}\int \mathcal{D}q\, e^{iS[q]}} \qquad (1.20)$$

with $S[q] = \int_{t_0}^{t} dt' \{ \underbrace{\tfrac{1}{2} m \dot{q}^2 - V(q)}_{L(q,\dot{q})} \}$

N: Normalisation, see eq. (1.16). It will be taken care of later.

Correlation functions

We repeat the analysis in the presence of further position operators: $t \geq t_n \geq \ldots \geq t_1 \geq t_0$

$$\langle q_f | \mathcal{U}(t,t_n)\, \hat{q}\, \mathcal{U}(t_n,t_{n-1}) \ldots \mathcal{U}(t_2,t_1)\, \hat{q}\, \mathcal{U}(t_1,t_0) | q_{in} \rangle$$

$$= \langle q_f, t | \ \hat{q}(t_n) \ldots \hat{q}(t_1) | q_{in}, t_0 \rangle \qquad (1.21)$$

Heisenberg pic.

with $\hat{q}(t) = \mathcal{U}(0,t)\, \hat{q}\, \mathcal{U}(t,0)$

$|q,t\rangle = \mathcal{U}(0,t)\, |q\rangle$

$$n=0: \quad \langle q_f, t | q_{in}, t_0 \rangle \simeq \int \mathcal{D}q \, e^{iS[q]} \qquad (1.22)$$

$n=1$: $\langle q_f, t | \hat{q}(t_1) | q_{in}, t_0 \rangle$

$$= \int dq \langle q_f, t | q, t_1 \rangle \ q \ \langle q, t_1 | q_{in}, t_0 \rangle$$

$$= \int dq \ q \ \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_1)=q}} e^{iS[q]} \int \mathcal{D}q \Big|_{\substack{q(t_1)=q \\ q(t_0)=q_{in}}} e^{iS[q]}$$

$$= \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_0)=q_{in}}} q(t_1)\, e^{iS[q]} \qquad (1.23)$$

in general: $t \geq t_n \geq \cdots \geq t_1 \geq t_0$

$$\langle q_f, t | \hat{q}(t_n) \cdots \hat{q}(t_1) | q_{in}, t_0 \rangle$$

$$= \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_0)=q_{in}}} q(t_n) \cdots q(t_1)\, e^{iS[q]} \qquad (1.24)$$

In most cases we are interested in vacuum amplitudes (see also QFT I, chapter)

$$\langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \qquad (1.25)$$

For the projection on the vacuum $|0\rangle$, $H|0\rangle = E_0|0\rangle$ the lowest energy state, we introduce a damping factor: (eq. (1.16), (1.17))

$$e^{-i\Delta t H} \to e^{-i\Delta t(1-i\varepsilon)H}$$
$$= e^{-i\Delta t H}\, e^{-\Delta t H \varepsilon} \qquad (1.26)$$

Higher energy states are suppressed by $e^{-(E-E_0)\Delta t \varepsilon}$ at each time step.

It follows that

$$\lim_{T\to\infty} \langle q_f, T | \hat{q}(t_n) \cdots \hat{q}(t_1) | q_{in}, -T \rangle$$
$$\simeq \langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \tag{1.27}$$

Remark: Eq. (1.27) holds for all states which overlap with the vacuum

$$\langle q_f, T | 0 \rangle \neq 0 .$$

Hence we have: $t_n \geq t_{n-1} \geq \cdots \geq t_1$

$$\langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle$$
$$= \int \mathcal{D}q \; q(t_n) \cdots q(t_1) \, e^{i S_\varepsilon[q]} \tag{1.28}$$

with
$$S_\varepsilon[q] = \int dt \left(\frac{1}{2} m \dot{q}^2 - V(q) + i \varepsilon q^2 \right) \tag{1.29}$$

Eq. (1.29) has the same effect as eq. (1.26).

The derivation of eq. (1.28) was done with $t_n \geq t_{n-1} \geq \dots \geq t_1$. The <u>path integral</u> or <u>functional integral</u> on the rhs of eq. (1.28) can be written down for arbitrary times $t_1, \dots, t_n$. Hence we have finally

$$\boxed{\begin{aligned} &\langle 0 | T\, \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \qquad (1.30) \\ &\quad \simeq \int \mathcal{D}q \; q(t_n) \cdots q(t_1)\, e^{iS[q]} \end{aligned}}$$

where the ε-prescription eq. (1.29) is understood.

Normalised correlation functions read

$$\boxed{\begin{aligned} &\frac{\langle 0 | T\, \hat{q}(t_1) \cdots \hat{q}(t_n) | 0 \rangle}{\langle 0 | 0 \rangle} \\ &\qquad = \frac{\int \mathcal{D}q \; q(t_1) \cdots q(t_n)\, e^{iS[q]}}{\int \mathcal{D}q \; e^{iS[q]}} \qquad (1.31) \end{aligned}}$$

Remarks:

(1) The quadratic part of the action S_ε reads ($m=1$)

$$S_\varepsilon[q] = \frac{1}{2}\int dt\; q(t)\left[-\partial_t^2 - \omega^2 + i\varepsilon\right]q(t) \tag{1.32}$$

The propagator that follows from eq. (1.32) is in momentum space (frequency space)

$$\frac{1}{p^2 - \omega^2 + i\varepsilon} = C \tag{1.33}$$

covariance

wich is the time-ordered (Feynman) propagator.

(2)* Changing S_ε in eq. (1.32), eq. (1.33) to path integrals that do <u>not</u> describe time-ordered products.

$$\boxed{\text{Choice of covariance } C \iff \text{choice of ordering}}$$

Generating Functional

The expectation values $\frac{\langle 0|T\hat{q}(t_1)\circ\circ\circ\hat{q}(t_n)|0\rangle}{\langle 0|0\rangle}$

are the normalised moments of the path integral $\int \mathcal{D}q\, e^{iS[q]}$.

Toy example: consider $\int_{\mathbb{R}} dq\, e^{iS(q)}$ with moments

$$\langle q^n\rangle := \frac{1}{N}\int_{\mathbb{R}} dq\, q^n\, e^{iS(q)} \qquad (1.34)$$

$\int dq\, e^{iS(q)}$

These moments are generated by

$$Z(j) = \frac{1}{N}\int dq\, e^{i\{S(q) + j\cdot q\}} \qquad (1.35)$$

$$\Rightarrow \langle q^n\rangle = (-1)^n \frac{\partial^n Z}{\partial j^n}$$

Gaußian theory: $S_0(q) = \frac{1}{2} q \alpha q$

$$\Rightarrow Z_0(j) = \frac{1}{N_0} \int_{\mathbb{R}} dq \, e^{i\{\frac{1}{2} q \alpha q + j q\}}$$

$$= \frac{1}{N_0} \int_{\mathbb{R}} dq \, e^{i\{\frac{1}{2}(q + \frac{1}{\alpha} j)^2 - \frac{1}{2} j \frac{1}{\alpha} j\}}$$

$$= e^{-\frac{i}{2} j \boxed{\frac{1}{\alpha}} j}$$

Covariance / Propagator

Moments:

$$\langle q^{2n} \rangle = \frac{2n!}{n!} \frac{1}{2^n} \frac{1}{\alpha^n}$$

Interacting theory: $S(q) = \frac{1}{2} q \alpha q - V(q)$

$$Z(j) = \frac{N_0}{N} e^{-i V(-i \frac{\partial}{\partial j})} Z_0(j)$$

Perturbation theory: expansion in powers of V

see exercise

QM: $q_j \to q_j(t)$; derivative $\to$ functional derivative

$$\boxed{\frac{\delta j(t)}{\delta j(t')} := \delta(t-t')} \qquad (1.36)$$

It follows that

$$\frac{\delta}{\delta j(t_1)} e^{i\int_{\mathbb{R}} dt\, j(t) q(t)} = i \int_{\mathbb{R}} dt \overbrace{\frac{\delta j(t)}{\delta j(t_1)}}^{\delta(t-t_1)} q(t)\, e^{i\int_{\mathbb{R}} \cdots}$$

$$= i\, q(t_1)\, e^{i\int_{\mathbb{R}} dt\, j(t) q(t)} \qquad (1.37)$$

$\Rightarrow$ Generating functional:

$$Z[j] = \frac{1}{N} \int \mathcal{D}q\, e^{i\{S[q] + \int dt\, j(t) q(t)\}} \qquad (1.38)$$

$$\text{with } N = \int \mathcal{D}q\, e^{iS[q]}$$

with

$$\langle 0 | T \hat{q}(t_1) \cdots \hat{q}(t_n) | 0 \rangle / \langle 0 | 0 \rangle$$

$$= (-i)^n \left. \frac{\delta^n Z[j]}{\delta j(t_1) \cdots \delta j(t_n)} \right|_{j=0} \qquad (1.39)$$

$F[j]$ functional of $j(t)$, e.g. $F[j] = \int_{\mathbb{R}} dt\, j(t)\, q(t)$

functional derivative: $\mathcal{D}_g F[j]$

$$F[j+\varepsilon g] = F[j] + \mathcal{D}_g F[j]\,\varepsilon + O(\varepsilon^2)$$

With $F[j] = \int_{\mathbb{R}} dt\; j(t)\, q(t)$

$$\Rightarrow \mathcal{D}_g F[j] = \int_{\mathbb{R}} dt\; g(t)\, q(t)$$

Eq. (1.36): $\mathcal{D}_g F[j] = q(t') \Rightarrow g_{t'}(t) = \delta(t-t')$

Also:

$$\mathcal{D}_g F[j] = \frac{\partial F[j+\varepsilon g]}{\partial \varepsilon}$$

$V''(0)$

Gaußian theory: $S_0[q] = \frac{1}{2}\int dt\, q(t)\left(-\partial_t^2 - \omega^2\right)q(t)$

see p. 11

Generating functional:

$$Z_0[j] = \frac{1}{N_0}\int \mathcal{D}q\; e^{i\{S[q] + \int dt\, j(t)\, q(t)\}}$$

$$= \frac{1}{N_0}\int \mathcal{D}q\; e^{iS[q']} \cdot e^{-\frac{1}{2}\int dt\, dt'\, q(t)\, G(t,t')\, q(t')} \qquad (1.40)$$

with

$$q'(t) = q(t) - i\int dt'\, G(t,t')\, j(t') \qquad (1.41)$$
$$\left(-\partial_t^2 - \omega^2\right) G(t,t') = i\delta(t-t')$$

The path integral measure is translation invariant:

$$\mathcal{D}q = \mathcal{D}q' \qquad (1.42)$$

which follows directly from the translation invariance of $dq = dq'$ and

$\mathcal{D}q = \prod_i dq(t_i)$, eq. (1.18), p. 6

Consequently we have

$$\boxed{Z_0[j] = e^{-\frac{1}{2}\int dt\, dt'\, q(t)\, G(t,t')\, q(t')}} \quad (1.43)$$

with the propagator $G(t,t')$. Note that the choice of $G(t,t')$ determines the path integral.

Interacting theory: $S[q] = S_0[q] - \int dt\, V(q)$

$$\boxed{Z[j] = \frac{N_0}{N} e^{-i\int dt\, V\left(-i\frac{\delta}{\delta j}\right)} Z_0[j]} \quad (1.44)$$

with moments

$$\boxed{\begin{aligned} &\langle 0|T\hat{q}(t_1)\cdots\hat{q}(t_n)|0\rangle / \langle 0|0\rangle \\ &\qquad = \left.\frac{\delta^n Z[j]}{\delta\hat{q}(t_1)\cdots\delta\hat{q}(t_n)}\right|_{j=0} \end{aligned}} \quad (1.45)$$