1.2 Functional integral for scalar fields

Consider now a scalar field in d dim.

$$\varphi = \varphi(x) \quad \text{with } x \in \mathbb{R}^d \qquad (1.46)$$

with d-1 spatial dimensions, $x = (x_0, \vec{x})$.

Action:

$$S[\varphi] = \int d^d x \underbrace{\left\{ \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right\}}_{\mathcal{L}\,:\,\text{Lagrange density}}$$

Hamiltonian:

$$H = \int d^{d-1}x \{ \underbrace{\pi(\vec{x})}_{= \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi}} \partial_0 \varphi - \mathcal{L} \} \qquad (1.47)$$

$$= \int d^{d-1}x \left\{ \frac{1}{2} \pi_0 \pi^0 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + V(\varphi) \right\}$$

As in QM we consider the transition amplitude

$$\underbrace{\langle \varphi_f}_{\text{final state}} | \mathcal{U}(t, t_0) | \underbrace{\varphi_{in} \rangle}_{\text{in state}} \qquad (1.48)$$

with states
$$\hat{\varphi}\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle$$
$$\hat{\Pi}\,|\Pi\rangle = \Pi(\vec{x})\,|\Pi\rangle \qquad (1.49)$$

Eq. (1.49) can be obtained in the many-body (field theory) limit of QM (see also QFT I).

$$\underset{\text{eq. (1.3), p. 1}}{\Psi(q)} \longrightarrow \Psi(q_1, \dots, q_n) \rightarrow \Psi[\varphi]$$
$$\delta(q-q') \longrightarrow \prod_i \delta(q_i - q_i') \rightarrow \delta[\varphi - \varphi'] \qquad (1.50)$$

Clean way: put theory on a space-time lattice

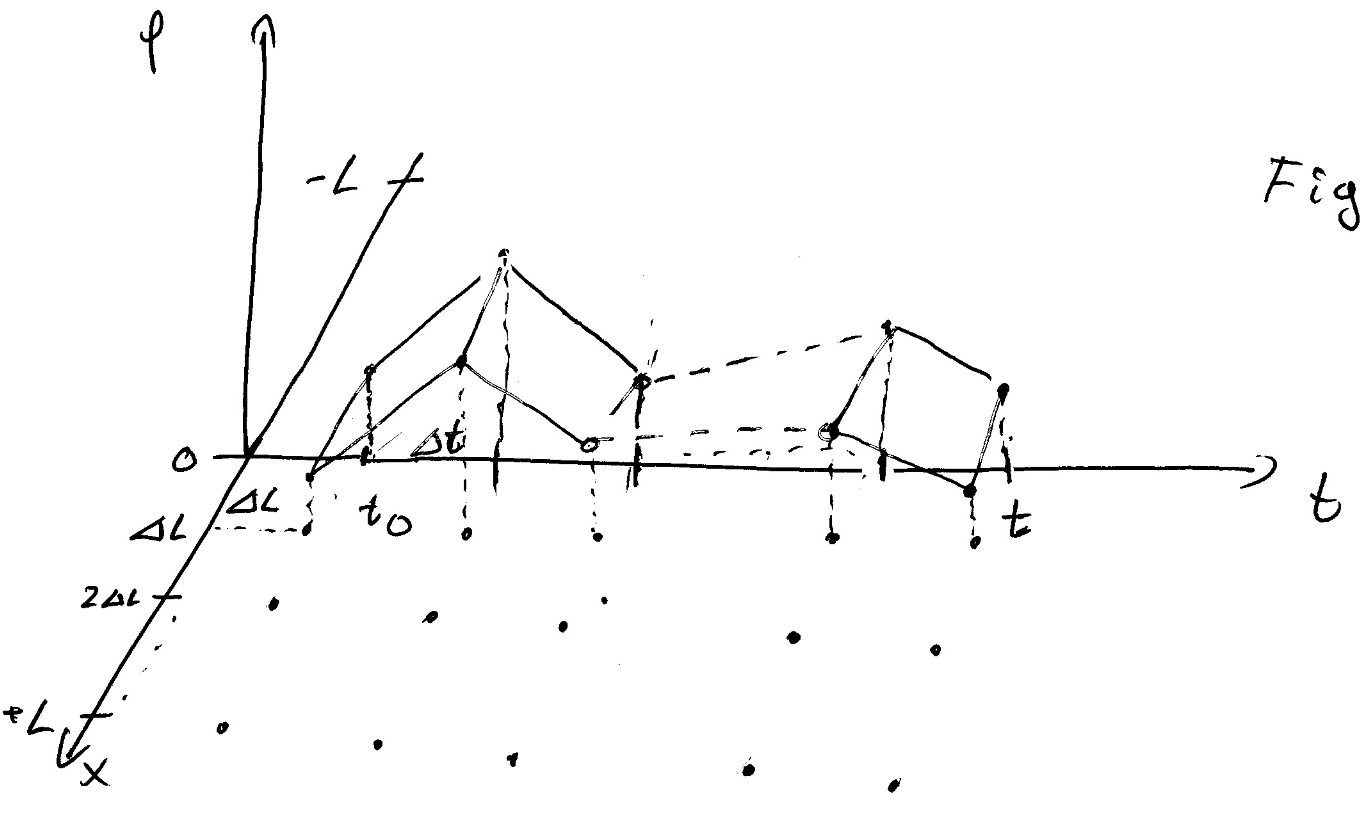

Fig. 1.2

Completeness relation:

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\, |\varphi\rangle\langle\varphi| \quad , \langle\varphi|\varphi'\rangle = \delta[\varphi(\vec{x}) - \varphi'(\vec{x})] \overset{\prod_{\vec{i}} \delta(\varphi_{\vec{i}} - \varphi'_{\vec{i}})}{=} \tag{1.51}$$

Repeating all steps in QM in the present field theoretical setting leads to

$$\boxed{\langle 0 | T\, \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) | 0 \rangle / \langle 0|0\rangle = \frac{\int \mathcal{D}\varphi\, \varphi(x_1) \cdots \varphi(x_n)\, e^{iS[\varphi]}}{\int \mathcal{D}\varphi\, e^{iS[\varphi]}}} \tag{1.52}$$

with

$$S = \int d^d x \Big\{ \underbrace{\tfrac{1}{2}\varphi(x)\big[-\Delta - m^2 + \overset{\text{time-ordering}}{i\varepsilon}\big]\varphi(x)}_{S_0} - V_{int}(\varphi) \Big\} \tag{1.53}$$

and

$$\boxed{\mathcal{D}\varphi = \prod_{\hat{i} \in \text{lattice}} d\varphi_{\hat{i}}} \tag{1.54}$$

$$\hat{i} = (i_0, \vec{i})$$

Alternative representations:

$$\varphi(x) = \sum_n c_n \varphi_n(x) \tag{1.54}$$

with $(\varphi_n, \varphi_m) = \delta_{nm}$ $\quad$ $\{\varphi_n\}$ orthonormal system

$$\overset{\shortparallel}{\int d^dx\, \varphi_n(x)\, \varphi_m(x)}$$

$$\Rightarrow \boxed{\mathcal{D}\varphi \simeq \prod_n dc_n} \tag{1.55}$$

Hence $\int \mathcal{D}\varphi\, e^{iS[\varphi]} \simeq \int \prod_n dc_n\, e^{iS[\varphi]}$

Example: free theory (Gaußian)

$$S[\varphi] = S_0[\varphi] \tag{1.56}$$

$\leftarrow$ eq. (1.53)

Choice of φ_n for (1.55):

$$\boxed{(-\Delta - m^2)\,\varphi_n = \lambda_n \varphi_n} \tag{1.57}$$

with $\lambda_n \in \mathbb{R}^+$.

It follows:

$$\boxed{\int \mathcal{D}\varphi \, e^{iS_0[\varphi]} \simeq \int \prod_n dc_n \, e^{\frac{i}{2}\sum_n c_n^2 \lambda_n}} \tag{1.58}$$

with $iS_0[\varphi] = \frac{i}{2} \sum_{n,m} \int d^dx \, c_n c_m \cdot \varphi_n(x) [-\Delta - m^2 + i\varepsilon] \varphi_m(x)$

drop $i\varepsilon$ $\rightarrow$

$$= \frac{i}{2} \sum_{n,m} c_n c_m \lambda_m \underbrace{\int d^dx \, \varphi_n(x) \varphi_m(x)}_{\delta_{nm}}$$

$$= \frac{i}{2} \sum_n c_n^2 \lambda_n \tag{1.59}$$

Finally:

$$\boxed{\begin{aligned} \int \mathcal{D}\varphi \, e^{iS_0[\varphi]} &\simeq \prod_n \lambda_n^{-1/2} \\ &= \det{}^{-1/2}(-\Delta - m^2) \end{aligned}} \tag{1.60}$$

The functional integral of the free theory gives us the $\det^{-1/2}$ of the kinetic operator (in the free theory).