1.3 Feynman rules

Generating functional:

$$\boxed{Z[J] = \frac{1}{N} \int \mathcal{D}\varphi \, e^{i\left\{S[\varphi] + \int d^dx \, J(x)\varphi(x)\right\}}} \qquad (1.61)$$

with

$$\langle 0 | T \hat{\varphi}(x_1) \ldots \hat{\varphi}(x_n) | 0 \rangle / \langle 0 | 0 \rangle$$

$$= (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \bigg|_{J=0} \qquad (1.62)$$

where we have used

$$\frac{\delta J(x)}{\delta J(y)} := \delta^d(x-y) \qquad (1.63)$$

For the free theory we have

$$\boxed{Z_0[J] = e^{-\frac{1}{2}\int d^dx \, d^dy \, J(x) \, G(x,y) \, J(y)}}$$

where $\left(-\Delta - m^2\right)_x G(x,y) = i\delta^d(x-y)$ (1.64)

Examples: (i) 2-point fct.

$$\langle T\varphi(x)\varphi(y)\rangle = -\left.\frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(y)}\right|_{J=0}$$

$$\shortparallel$$

$$\langle 0|T\hat{\varphi}(x)\,\hat{\varphi}(y)|0\rangle / \langle 0|0\rangle$$

$$= \frac{\delta^2}{\delta J(x)\,\delta J(y)}\int d^dx'\,d^dy'\,J(x')\,G(x',y')\,J(y')$$

$$= \underline{G(x,y)} \qquad (1.65)$$

(ii) 4-point fct.:

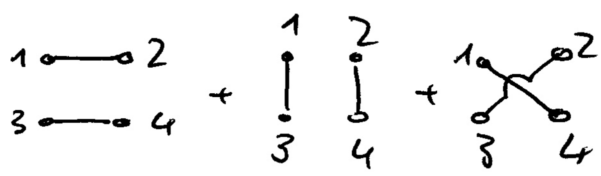
$$\langle T\varphi(x_1)\cdots\varphi(x_4)\rangle = \left.\frac{\delta^4 Z_0[J]}{\delta J(x_1)\cdots\delta J(x_4)}\right|_{J=0}$$

$$= G(x_1,x_2)\,G(x_3,x_4)$$

$$+ \text{permuts} \qquad (1.66)$$

Diagrammatically:

$$\langle T\varphi(x_1)\cdots\varphi(x_4)\rangle =$$



(iii) General $2n$-point fct.:

$$\langle T\,\varphi(x_1)\cdots\varphi(x_{2n})\rangle = (-i)^{2n}\frac{\delta^{2n} Z[J]}{\delta\varphi(x_1)\cdots\delta\varphi(x_{2n})}\Bigg|_{J=0}$$

$$= \frac{(-i)^{2n}}{n!}\,\frac{\delta^{2n}}{\delta\varphi(x_1)\cdots\delta\varphi(x_{2n})}\,\frac{(-1)^n}{2^n}\left[\int d^dx\,d^dy\,J(x)G(x,y)J(y)\right]^n$$

$$= G(x_1,x_2)\,G(x_3,x_4)\cdots G(x_{2n-1},x_{2n})$$

$$+ \text{permuts.} \qquad (1.67)$$

Schwinger functional:

$$\boxed{W[J] = \ln Z[J]} \qquad (1.68)$$

In the free case:

$$W_0[J] = \ln Z_0[J] = -\frac{1}{2}\int d^dx\,d^dy\,J(x)G(x,y)\cdot J(y)$$

(1.69)

and hence

$$(-i)^2\frac{\delta^2 W_0[J]}{\delta J(x_1)\,\delta J(x_2)}\Bigg|_{J=0} = G(x_1,x_2) \qquad (1.70)$$

In terms of Z_0 this reads

$$\left.\frac{\delta^2 W_0[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = \left(\frac{1}{Z_0[J]}\frac{\delta^2 Z_0}{\delta J(x_1)\delta J(x_2)} - \frac{1}{Z_0}\frac{\delta Z_0}{\delta J(x_1)}\,\frac{1}{Z_0}\frac{\delta Z_0}{\delta J(x_2)}\right)_{J=0} \tag{1.71}$$

$$= \underbrace{\langle T\,\varphi(x_1)\,\varphi(x_2)\rangle - \langle\varphi(x_1)\rangle\langle\varphi(x_2)\rangle}_{\text{connected 2-point fct.}} \quad (1.71)$$

Here we have $\langle\varphi(x_1)\rangle = \left.\int dx\, G(x_1,x)\,J(x)\right|_{J=0} = 0$

$$(1.72)$$

Interacting theory: $S[\varphi] = S_0[\varphi] - \int d^dx\, V_{int}(\varphi)$

$$Z[J] \simeq \int \mathcal{D}\varphi\, e^{i\left\{S_0[\varphi] - \int d^dx\, V_{int}(\varphi) + \int d^dx\, J(x)\varphi(x)\right\}}$$

$$= e^{-i\int d^dx\, V_{int}\left(-i\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\varphi\, e^{iS_0[\varphi] + \int d^dx\, J(x)\varphi(x)\}}$$

$$\simeq e^{-i\int d^dx\, V_{int}\left(-i\frac{\delta}{\delta J(x)}\right)}\, Z_0[J] \tag{1.73}$$

see also QM, eq. (1.44), p. 15.

Using eq. (1.64), p. 21 for Z_0, we get for eq. (1.73):

$$\boxed{Z[J] \simeq e^{-i\int d^dx V_{int}\left(-i\frac{\delta}{\delta J(x)}\right)} e^{-1/2\int_{x,y} J(x) G(x,y) J(y)}} \quad (1.74)$$

with $\int_x = \int d^dx$, $\int_{x,y} = \int d^dx d^dy$.

Asume for the moment, that $V_{int}(.)$ is small. Then we can expand Z in powers of V: $J \cdot G \cdot J = J(x)\, G(x,y)\, J(y)$

$$Z[J] \simeq \left[1 - i\int_x V_{int}\left(\frac{\delta}{\delta J}\right) - \frac{1}{2}\left(\int_x V_{int}\left(\frac{\delta}{\delta J}\right)\right)^2 + \dots\right] Z_0[J]$$

$$= \left[1 - i\int_x V_{int}\left(\frac{\delta}{\delta J}\right) + \dots\right]\left(1 - \frac{1}{2}\int_{x,y} J\cdot G\cdot J + \frac{1}{8}\left(\int_{x,y} J\cdot G\cdot J\right)^2 + \dots\right) \quad (1.75)$$

In φ^4-theory we have $V_{int}(\varphi) = \frac{\lambda}{4!}\varphi(x)^4$,

$$\boxed{i\int_x V_{int}\left(\frac{\delta}{\delta J}\right) = \frac{i\lambda}{4!}\int_x \left(\frac{\delta}{\delta J(x)}\right)^4} \quad (1.76)$$

Diagrammatics in φ^4-theory:

(i) Vacuum 'physics': What is $Z[0]$?

$$Z[0] \simeq e^{-i\int_x V_{int}\left(\frac{\delta}{\delta J}\right)} e^{-1/2\int_{x,y} J\cdot G\cdot J}\Big|_{J=0}$$

$$= 1 - i\int_x V_{int}\left(\frac{\delta}{\delta J}\right)\frac{1}{2}\left(-\frac{1}{2}\int_{x,y} J\cdot G\cdot J\right)^2 \quad (\alpha)$$

$$-\frac{1}{2}\left[\int_x V\left(\frac{\delta}{\delta J}\right)\right]^2 \frac{1}{4!}\left(-\frac{1}{2}\int_{x,y} J\cdot G\cdot J\right)^4 \quad (\beta)$$

$$+ \dots \qquad (1.77)$$

(a) Each J-derivative can hit each current J.

(b) Due to the symmetry of G: $G(x,y) = G(y,x)$

we have $\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{1}{2}\int_{x,y} J\cdot G\cdot J = G(x_1,x_2)$.

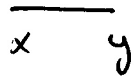
$\Rightarrow$ (c) We simply have to work-out all permutations

$\Rightarrow$ best done diagrammatically

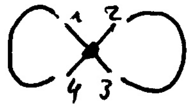
(or computer)

Diagrams:

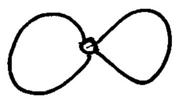
$$-\frac{i\lambda}{4!}\int_x \left(\frac{\delta}{\delta J(x)}\right)^4 \sim$$  vertex: $-i\lambda$

(b) $\rightarrow$ $$-\frac{1}{2}\int_{x,y} j(x)\, G(x,y)\, j(y) \sim$$  propagator

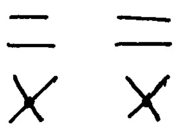
'Computation' of $Z[0]$: eq. (1.77)

α : $\frac{1}{8}\,\frac{1}{4!}$  $\cdot 4!$ ← from taking derivatives

permutations of attaching vertex legs with props.

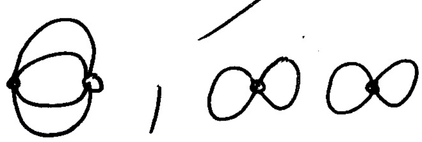
$= \frac{1}{8}$  $= -\frac{i\lambda}{8}\int d^dx\; G(x,x)\, G(x,x)$

divergent, see QFT I

β : $\frac{1}{4!}\,\frac{1}{2^4}\,\frac{1}{2}\left(\frac{1}{4!}\right)^2$ 

$= \frac{1}{4!}\,\frac{1}{2^4}\,\frac{1}{2}\left(\frac{1}{4!}\right)^2$ comb.  $+ \ldots$

(exercise)



Remark: What about $Z_0[0]$?

We use

$$\int \mathcal{D}\varphi \, e^{iS_0[\varphi]} \simeq \det{}^{-1/2}(-\Delta - m^2)$$

$$\simeq \det{}^{1/2} G \qquad (1.78)$$

Furthermore we have

$$\boxed{\det M = \prod_n \lambda_n = \prod_n e^{\ln \lambda_n} = e^{\sum_n \ln \lambda_n} = e^{\operatorname{Tr} \ln M} \qquad (1.79)}$$

For $Z_0[0]$ this implies

$$Z_0[0] \simeq \det{}^{1/2} G = e^{\frac{1}{2}\operatorname{Tr} \ln G} \qquad (1.80)$$

and $$\operatorname{Tr} \ln G = \int d^d x \, \ln G(x,x)$$

or diagrammatically (with $\partial_{m^2} G = i G^2$)

$$\frac{1}{Z_0[0]} \partial_{m^2} Z_0[0] \simeq \frac{i}{2} \bigcirc \qquad (1.81)$$

Correlation fcts.:

$$\langle T\varphi(x_1)\cdots\varphi(x_n)\rangle = \frac{(-i)^n}{Z[0]}\frac{\delta^n Z[J]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\Bigg|_{J=0}$$

$$= \frac{(-i)^n}{1+\text{vac. diagr.}}\frac{\delta^n}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\, e^{-i\int_x V_{int}\left(\frac{\delta}{\delta J}\right)} e^{-\frac{1}{2}\int_{x,y} J\cdot G\cdot J}$$

$$= (-i)^n\left[\frac{\delta^n}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\, e^{-i\int_x V_{int}\left(\frac{\delta}{\delta J}\right)} e^{-\frac{1}{2}\int_{x,y} J\cdot G\cdot J}\right]_{\text{no vac. contr.}}$$

(1.82)

$\Rightarrow$ Feynman rules:

Propagator: x —— y $= G(x,y)$

Vertex:  $= -i\lambda\int d^4x$

(a) Write down all diagrams in a given order N of λ of $2n$-point correlation fct.

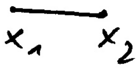
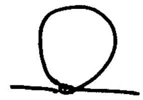
(b) Combinatorical factors of diagrams

$$\left(\frac{1}{4!}\right)^N \frac{1}{N!}\frac{1}{(2N+n)!}\frac{1}{2^{2N+n}}\ (\text{permut.})$$

$$\sum_N \left(\frac{1}{4!}\right)^N \frac{1}{N!}\left[\int_x \frac{\delta}{\delta J}\right]^N \frac{1}{(2N+n)!}\left[\frac{1}{2}\int_{x,y} J\cdot G\cdot J\right]^{2N+n}$$

$\llcorner N$ loops

Example: 2-point fct.

$$\langle T\, \varphi(x_1)\, \varphi(x_2) \rangle =$$  $+ \frac{1}{2}$  $+ \dots$

$\frac{1}{3!}\ \frac{1}{8}\ \frac{1}{4!} \cdot 4!\ \ 4 \cdot 6$

$\frac{1}{3!}\frac{1}{2^6}(JGJ)^3 \quad \frac{\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4 \quad \frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}$

The above results extend straightforwardly to general scalar theories.

Example: $S[\varphi, \phi] = \frac{1}{2} \int_x \left\{ \varphi(-\Delta - m_\varphi^2)\varphi + \phi(-\Delta - m_\phi^2)\phi \right\}$

$$- \frac{1}{2} \int_x \phi\, \varphi^2 \qquad (1.83)$$

Generating functional:

$$Z[J_\varphi, J_\phi] = \int \mathcal{D}\varphi \mathcal{D}\phi\, e^{i\left\{ S[\varphi,\phi] + \int_x (J_\varphi \varphi + J_\phi \phi) \right\}}$$

$$= e^{-\frac{i}{2}\int_x \frac{\delta}{\delta J_\phi} \frac{\delta^2}{\delta J_\varphi^2}}\, Z_0[J_\varphi, J_\phi]$$

with $\qquad (1.84)$

$$Z_0[J_\varphi, J_\phi] = e^{-\frac{1}{2}\int_{x,y} \left(J_\varphi G_\varphi J_\varphi + J_\phi G_\phi J_\phi \right)}$$

$$(-\Delta_x - m^2_{\varphi/\phi})\, G_{\varphi/\phi}(x,y) = i\,\delta(x-y) \qquad (1.85)$$

Wick rotation & statistical interpretation

The functional integrals are defined with measures $d\varphi\, e^{iS[\varphi]}$. For practical purposes (perturbative evaluation, numerics) it is convenient and necessary, to perform a Wick rotation (see also QFT I):

$$\boxed{x_{0_M} \to -i\, x_{0_E}} : \boxed{e^{-i\Delta t H} \to e^{-\Delta t H}}$$

$$\Rightarrow \quad i \int d^d x_M \to + \int d^d x_E \qquad (1.86)$$

$$- \partial_\mu \partial^\mu \to + \partial_\mu \partial_\mu$$

$$\varphi \text{ scalar}: \quad \varphi \to \varphi$$

$$\Rightarrow \quad \overbrace{i \int d^d x \left\{ \frac{1}{2} \varphi (-\partial_\mu \partial^\mu) \varphi - V(\varphi) \right\}}^{S[\varphi]\ \text{Minkowski}} \qquad (1.87)$$

$$\to - \underbrace{\int d^d x \left\{ \frac{1}{2} \varphi (-\partial_\mu \partial_\mu) \varphi + V(\varphi) \right\}}_{S[\varphi]\ \text{Euclidean}}$$

$$\boxed{S[\varphi] \geq 0}$$

Generating Functional:

$$\boxed{Z_{\text{Euclidean}}[J] = \int \mathcal{D}\varphi \; e^{-S[\varphi] + \int d^dx\, J(x)\,\varphi(x)} \qquad (1.88)}$$

with $\langle T\, \varphi(x_1) \cdots \varphi(x_n) \rangle_{\text{Eucl.}} = \dfrac{\delta^n Z}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}$ (1.89)

Free theory: $S_0[\varphi] = \frac{1}{2}\int d^dx\, \varphi(-\underset{\Delta_E = \partial_\nu\partial_\nu}{\Delta_E} + m^2)\varphi$

$$Z_0[J] = \int \mathcal{D}\varphi\, e^{-S_0[\varphi] + \int d^dx\, J(x)\,\varphi(x)}$$

$$= \det{}^{-1/2}\underbrace{(-\Delta + m^2)}_{\geq 0}\; e^{\frac{1}{2}\int d^dx\, J(x)\, G(x,y)\, J(y)} \qquad (1.90)$$

with $(-\Delta + m^2)\, G(x,y) = \delta^d(x-y)$ (1.91)

Wick rotation on Propagators

$$G_M(p) = \frac{i}{p_M^2 - m^2 + i\varepsilon} \longrightarrow G_E(p) = \frac{1}{p^2 + m^2}$$

Vertex

$$-i\lambda \longrightarrow -\lambda$$

The Euclidean functional integral is a statistical integral with statistical measure $d\varphi\, e^{-S[\varphi]}$ (not normalised).

The Schwinger functional is nothing but the free energy F.

Euclidean correlation fcts. can be Wick-rotated back to Minkowski space (or real time), where the provide the real-time correlation fcts.:

$$\boxed{\begin{aligned} &\langle T\, \varphi(x_1)_E \cdots \varphi(x_n)_E \rangle_{Eucl.}(x_{i_{0E}} \to i\, x_{i_{0M}}) \\ &= \langle T\, \varphi(x_1)_M \cdots \varphi(x_n)_M \rangle_{Mink.} \end{aligned}} \tag{1.92}$$