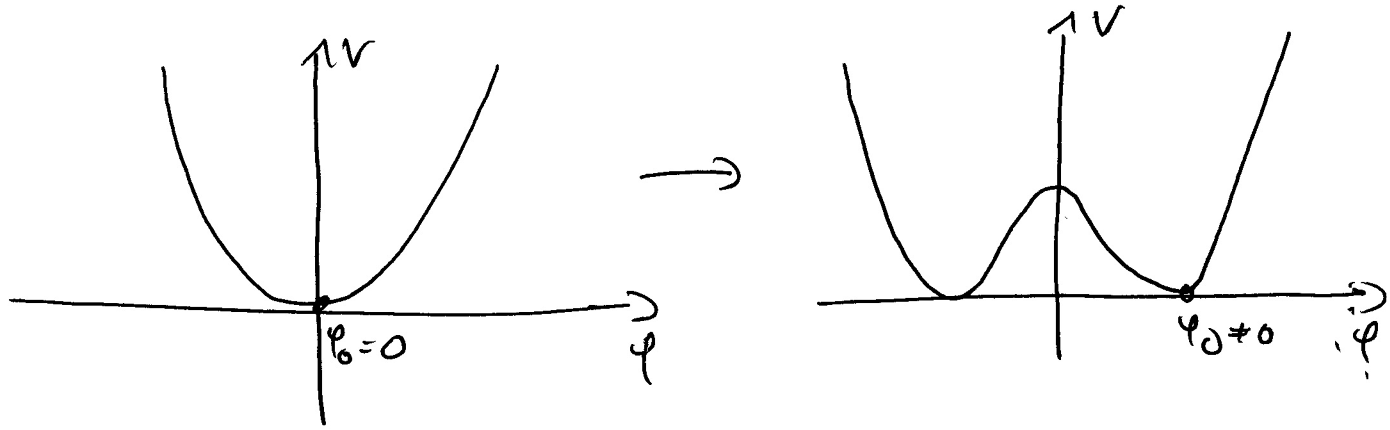
## 1.4 Effective action & spontaneous symmetry breaking

Coleman, Weinberg
PRD 7 (1973) 1888-1910
~4 kilo citations

In the first lecture course QFT I we have briefly discussed spontaneous symmetry breaking, e.g.




Here, we want to use the functional integral for computing the effective potential at one loop: $V_{\text{classical}} \to V_{\text{eff}}$.

Consider the generating functional $Z[J]$,

$$Z[J] \simeq \int d\varphi\, e^{-S[\varphi]+\int d^dx\, J(x)\varphi(x)} \tag{1.93}$$

with $J$ such that

$$\boxed{\phi := \langle\varphi\rangle_J = \frac{\delta}{\delta J} W[J]\Big|_{J_\phi}} \tag{1.94}$$

Now we compute $Z[J]$ in a <u>saddle point expansion</u> about the minimum $\phi$ the exponent in eq.(1.93),

$$\left.\frac{\delta S}{\delta \varphi}\right|_{\phi} = J \tag{1.95}$$

and hence

$$S[\varphi] - \int_x J\cdot\varphi = S[\phi] + \frac{1}{2}\int_{x,y} \varphi\cdot S^{(2)}[\phi]\cdot\varphi + O(\varphi^3) + \int_x J\,\phi \tag{1.96}$$

with

$$S^{(n)}[\phi](x_1,\ldots,x_n) = \frac{\delta^n S[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \tag{1.97}$$

Dropping the higher order terms in eq. (1.99), we get Legendre trafo of is

$$\Gamma_{1\text{-Loop}}[\phi] := \int_x J\cdot\phi - W \simeq -\ln\int d\varphi\, e^{-\frac{1}{2}\int_{x,y}\varphi\cdot S^{(2)}[\phi]\cdot\varphi} + S[\phi] \tag{1.98}$$

which gives us, by using eq.(1.80) with $(-\Delta + m^2) \to S^{(2)}[\phi]$:

$$\boxed{\Gamma_{1\text{-loop}}[\phi] = S[\phi] + \frac{1}{2}\,\mathrm{Tr}\ln S^{(2)}[\phi]} \tag{1.99}$$

Example: $\varphi^4$-theory in 3d/4d (Euclidean)

$$S[\varphi] = \int d^dx \left\{ \frac{1}{2}\varphi\cdot(-\Delta + m_0^2)\cdot\varphi + \underbrace{\frac{\lambda_0}{4!}\varphi^4}_{V_0(\phi)} \right\} \tag{1.100}$$

and

$$S^{(2)}[\phi](x,y) = \Big(-\Delta + \overbrace{m_0^2 + \frac{\lambda_0}{2}\phi^2}^{V_0''(\phi)}\Big)\,\delta^d(x-y)$$

For constant $\phi$ we have: $S^{(2)}[\phi](p,q) = S^{(2)}[\phi](p)(2\pi)^d\delta^d(p-q)$

$$S^{(2)}[\phi](p) = p^2 + m_0^2 + \frac{\lambda_0}{2}\phi^2 \tag{1.101}$$

This gives us for the (1-loop) effective pot. $\boxed{V_{\text{eff}} = \frac{1}{\underbrace{\int d^dx}_{\text{vol.}}}\Gamma[\phi]\Big|_{\phi\ \text{const}}}$

$$\boxed{V_{\text{eff}}(\phi) = V_0(\phi) + \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\ \ln\left(p^2 + m_0^2 + \frac{\lambda_0}{2}\phi^2\right) + \ln N} \tag{1.102}$$

We compute $V_{\text{eff}}$ by taking its $m_0^2$-derivative:

$$\partial_{m_0^2}V_{\text{eff}}(\phi, m_0^2) = \partial_{m_0^2}V_0(\phi) + \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\left[\frac{1}{p^2+m_0^2+\frac{\lambda_0}{2}\phi^2} - \frac{1}{p^2+m_0^2}\right] \tag{1.103}$$

Choose $\partial_{m_0^2}N = -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+m_0^2}$

Eq. (1.103) is the one-loop approx. to the (unrenormalised) Callan-Symanzik equation, that governs $\partial_{m_0^2}\Gamma$.

In 3d, eq. (1.103) is finite. Upon momentum integration we get

$$\partial_{m_0^2} V_{\text{eff}}(\phi; m_0^2) = \partial_{m_0^2} V_0(\phi) - \frac{1}{8\pi}\left\{\sqrt{m_0^2 + \lambda/2\,\phi^2} - \sqrt{m_0^2}\right\} \tag{1.104}$$

This finiteness relates to the fact that $\partial_{m_0^2}$  is finite in 3d.

Using $\partial_{\phi^2} V_{\text{eff}} = \frac{\lambda}{2} \partial_{m_0^2} V_{\text{eff}}$, we have $V_{\text{eff}} \simeq \frac{\lambda}{2} \int d\phi^2\, \partial_{m_0^2} V_{\text{eff}}$ and hence

$\lambda$ does not get renormalised in 3d

$$\boxed{V_{\text{eff}}(\phi) \simeq V(\phi) - \frac{1}{12\pi}\left(m^2 + \frac{\lambda}{2}\phi^2\right)^{3/2} + \lambda m \phi^2 \text{-correct.}} \tag{1.105}$$

renormalised

Remark: The nth field derivatives of $V_{\text{eff}}(\phi)$ at $\phi = 0$ give the amputated n-point Green functions at one loop.

(Exercise: Prove this for $n = 2, 4$)

In $d=4$, eq. (1.103) is not finite. We either take a further $m_0^2$-derivative, or we use the standard renormalisation with help of the bare parameters:

$$(0)\qquad \mathcal{N} = -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,\ln\left(p^2+m^2\right)\qquad V_{eff}(0)=0$$

(convenience)

$$\text{mass renorm. }(1)\qquad m_0^2 = m^2 - \frac{\lambda}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2}\quad \to \partial_\phi^2 V_{eff}\Big|_{\phi=0} = m^2$$

$$\text{coupl. renorm. }(2)\qquad \lambda_0 = \lambda + \frac{3}{2}\lambda^2\left[\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^2} + \frac{2}{3}c(\mu/m)\right]$$

$c$ such that $\partial_\phi^4 V_{eff}\Big|_{\phi=\mu} = \lambda$

[for $m\neq 0$ e.g. $\mu=0$ : $c=0$]

(1.106)

Inserting eq. (1.106) in eq. (1.102) we get

$$\boxed{V_{eff}(\phi) = V_{ren}(\phi) + \frac{1}{4!}c\,\lambda^2\phi^4 + \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\left\{\ln\left(1+\frac{\lambda/2\,\phi^2}{p^2+m^2}\right) - \frac{\lambda/2\,\phi^2}{p^2+m^2} + \frac{1}{2}\left(\frac{\lambda/2\,\phi^2}{p^2+m^2}\right)^2\right\}}\qquad(1.107)$$

with $V_{ren}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$

Upon momentum integration this yields

$$\boxed{V_{eff}(\phi) = V_{ren}(\phi) + \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda}{2}\phi^2\right)^2 \ln\left(1 + \frac{\lambda/2\,\phi^2}{m^2}\right) - \frac{1}{64\pi^2}\left(\frac{1}{2}m^2\phi^2 - \frac{3}{2}\left(\lambda/2\,\phi^2\right)^2\right) + \frac{1}{4!}c\lambda^2\phi^4} \tag{1.108}$$

$V_{eff}$ obeys the renormalisation conditions in (eq. 1.106) : $\partial_\phi^2 V_{eff}\big|_{\phi=0} = m^2$ & $\partial_\phi^4 V_{eff}\big|_{\phi=\mu} = \lambda$.

The latter is obtained for $m \neq 0$ e.g. for $\mu = 0$, $c = 0$.

Massless limit : $m \to 0$ (Coleman-Weinberg) $[\mu = M]$

$$\lim_{m\to 0} \partial_\phi^4 V_{eff}\Big|_{\phi=\mu} = \lambda \ : \quad c = -\frac{1}{4\pi} - \frac{3}{32\pi^2}\ln\left(1 + \lambda/2\,\mu^2/m^2\right) \tag{1.109}$$

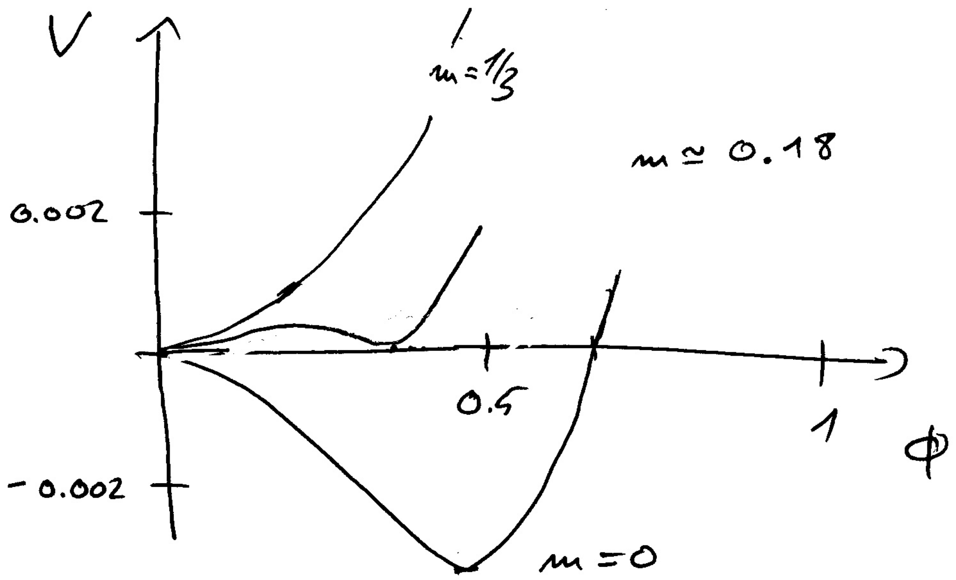
with

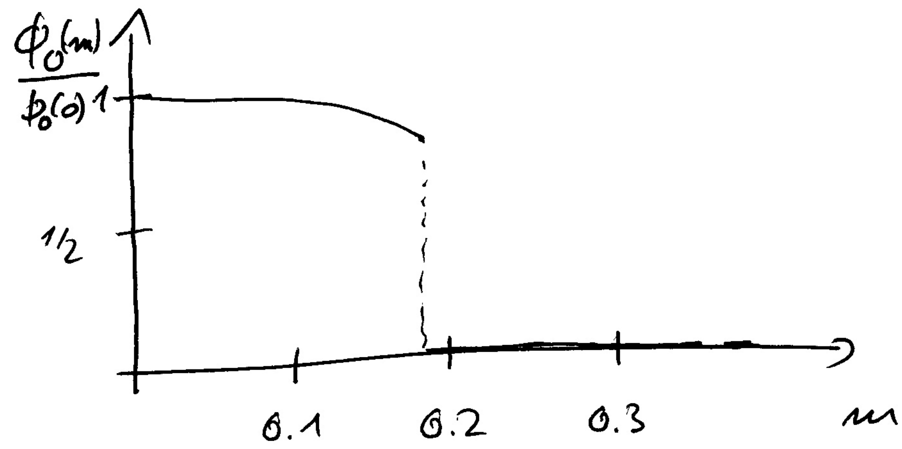
$$\boxed{V_{eff}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left[\ln\frac{\phi^2}{\mu^2} - \frac{25}{6}\right]} \tag{1.110}$$

$V_{eff}(\phi)$ has a non-trivial minimum at

$$\boxed{\begin{aligned} \ln\frac{\phi_0^2}{\mu^2} &= -\frac{32\pi^2}{3} + \frac{11}{3}\lambda \\ \Rightarrow \phi_0 &= \mu\, e^{-\frac{1}{3\lambda}\left(16\pi^2 - \frac{11}{2}\lambda\right)} \end{aligned}} \tag{1.11}$$

— $\lambda = 20$, $\mu = 1$ : $\phi_0 \simeq 0.45$ .. (measured in $\mu$)




Remarks: (1) 1st order phase transition

(2) beyond validity of perturb. theory!