2.2 Quantum field theory

As in QFT I we consider a

-Euclidean-Dirac action: (see QFT I, chapter 4)

$$S_D[\psi, \bar\psi] = -\int d^dx\, \bar\Psi(\not{\partial} + m)\psi \qquad (2.30)$$

no i

with

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}} \qquad (2.31)$$

hermitian representation

γ_5 or better γ_{2n+1} in even dimensions $d=2n$:

$$\boxed{\gamma_{2n+1} = \gamma_0\gamma_1 \cdots \gamma_{2n-1} \Rightarrow \gamma_{2n+1}^2 = \mathbb{1}} \qquad (2.32)$$

Spin-generators: (hermitian)

$$\boxed{\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]} \qquad (2.33)$$

generates $SU(2)\times SU(2)$ rotations in $d=4$.

Remark: Wick rotation: $x_{0M} \to -i x_{0E}$

$$\Rightarrow \gamma_{0M} \to -i\gamma_{0E}$$

$$\Rightarrow \bar{\psi}_M \to -i\bar{\psi}_E , \tag{2.34}$$

leading to eq. (2.30).

We get the functional integral in analogy

$$\boxed{Z[\eta,\bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_D[\psi,\bar{\psi}] + \int_x (\bar{\eta}\psi - \bar{\psi}\eta)}} \tag{2.35}$$

and

$$\boxed{\langle T\psi(x_1)\cdots\psi(x_n)\bar{\psi}(x_{n+1})\cdots\bar{\psi}(x_{2n})\rangle = \frac{1}{Z}\frac{\delta^{2n} Z}{\delta\bar{\eta}(x_1)\cdots\delta\eta(x_{2n})}\Bigg|_{\eta,\bar{\eta}=0}} \tag{2.36}$$

Interacting theories with fermions are e.g.

Yukawa-type: $\int_x \bar{\psi}(\not{\partial} + m + h\phi)\psi$ (scalar field; h: Yukawa coupl.)

QED: $\int_x \bar{\psi}(\not{\partial} - ie\not{A} + m)\psi$ (A: U(1) gauge field)

QCD: $\int_x \bar{\psi}(\not{\partial} - ig\not{A} + m)\psi$ (A: SU(N) gauge field)

Perturbation theory: expansion about Gaußian theory

$\Rightarrow$ Free fermionic generating functional:

$$Z_0[\eta,\bar\eta] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-S_0[\psi,\bar\psi]+\int_x(\bar\eta\psi-\bar\psi\eta)}$$

with (2.37)

$$S_0[\psi,\bar\psi] = \int_x \bar\psi(\partial\!\!\!/ + m)\psi$$

We define

$$\psi = \psi' - \frac{1}{\partial\!\!\!/+m}\cdot\eta$$
$$\bar\psi = \bar\psi' + \bar\eta\cdot\frac{1}{\partial\!\!\!/+m} \qquad (2.38)$$

and get from eq. (2.37) with eq. (2.38)

$$\boxed{Z_0[\eta,\bar\eta] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-S_0[\psi,\bar\psi]} \cdot e^{\int_{x,y}\bar\eta(x)\, G_\psi(x,y)\,\eta(y)}} \qquad (2.39)$$

with

$$(\partial\!\!\!/^x + m)\, G_\psi(x,y) = \delta^d(x-y) \qquad (2.40)$$

Fermionic determinant:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_0[\psi,\bar{\psi}]} = \det(\not\partial + m) \qquad (2.41)$$

In summary:

$$\boxed{Z_0[\eta,\bar{\eta}] = \det(\not\partial + m)\, e^{-\int_{x,y} \bar{\eta}\, G_\psi(x,y)\, \eta(y)} \qquad (2.42)}$$

Example: 2-point fct.

$$\langle T\psi(x)\, \bar{\psi}(y)\rangle = \frac{1}{Z} \frac{\delta^2}{\delta\bar{\eta}(x)\,\delta\eta(y)} Z \Big|_{\eta,\bar{\eta}=0}$$

$$= G_\psi(x,y) \qquad (2.43)$$

Interacting theories: eg.

$$S[\psi,\bar{\psi},\varphi] = S_D[\psi,\bar{\psi},\varphi] + S_\varphi[\varphi]$$

with $\quad S_D[\psi,\bar{\psi},\varphi] = \int_x \bar{\psi}(\not\partial + m + h\varphi)\psi$

$$(2.44)$$

$$\Rightarrow Z[\eta,\bar\eta,J] = \int \mathcal{D}\varphi \overbrace{\left[\int \mathcal{D}\psi \mathcal{D}\bar\psi\, e^{-S_D[\psi,\bar\psi,\varphi] + \int_x (\bar\eta\psi - \bar\psi\eta)} \right]}^{\text{Gaußian in fermion}} \cdot e^{-S_\phi[\varphi] + \int_x J\cdot\varphi} \tag{2.45}$$

Gaußian fermionic integration

$$\Rightarrow Z[\eta,\bar\eta,J] = \int \mathcal{D}\varphi\, e^{-S_{eff}[\varphi] + \int_x J\varphi} \cdot e^{\int_{x,y} \bar\eta \frac{1}{\not\partial + m + h\varphi} \eta}$$

with (2.46)

$$S_{eff}[\varphi] = S_\phi[\varphi] - \mathrm{Tr}\ln(\not\partial + m + h\varphi)$$

In eq. (2.46) we have used the Gaußian integration as for Z_0 (take $m+h\varphi$ as the effective mass).

Note that $-\mathrm{Tr}\ln(\not\partial + m + h\varphi)$ is nothing but the fermionic part of the one-loop effective potential, see eq. (1.99), p. 34.

Note the relative minus sign, which signals the fermionic loops.

Remarks:

(1) Hubbard-Stratonovich transformation

Assume (2.46) with scalar action

$$S_\varphi[\varphi] = \frac{1}{2} m^2 \varphi^2$$

$$\Rightarrow \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\int_x m^2 \varphi^2 + h\int_x \bar{\psi}\varphi\psi}$$

$$\simeq e^{-\frac{1}{2} h^2/m^2 \int_x (\bar{\psi}\psi)^2} \qquad (2.45a)$$

$$\Rightarrow Z[\eta,\bar{\eta}] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{+\int_x \bar{\psi}\not{\partial}\psi - \lambda_\psi \int_x (\bar{\psi}\psi)^2} \qquad (2.45b)$$

4-Fermi theory with coupling $\lambda_\psi = \frac{1}{2}\frac{h^2}{m^2}$

(2) Fermionic part of effective potential

$$-\partial_m \operatorname{Tr} \ln(\not{\partial} + m + h\varphi) = -\int \frac{d^d p}{(2\pi)^d} \operatorname{tr} \frac{1}{i\not{p} + m + h\varphi}$$

Dirac trace

$$= -\int \frac{d^d p}{(2\pi)^d} \operatorname{tr} \frac{-i\not{p} + m + h\varphi}{p^2 + (m+h\varphi)^2}$$

$$= -4(m+h\varphi)\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + (m+h\varphi)^2} \qquad (2.46a)$$

see (eq. 1.103), p. 35 with $m_0^2 + \lambda_0/2\, \phi^2 \to (m+h\varphi)^2$.

Perturbation theory: $S_\phi = \int_x \left(\frac{1}{2}\varphi(-\Delta + m_\phi^2)\varphi + V_{int}(\varphi)\right)$

$$Z[\eta, \bar{\eta}, J] = e^{-\int_x V\left(\frac{\delta}{\delta J}\right) + h\int_x \frac{\delta}{\delta \eta}\frac{\delta}{\delta J}\frac{\delta}{\delta \bar{\eta}}} \cdot e^{\frac{1}{2}\int_{x,y} J\cdot G_\phi J - \int_{x,y}\bar{\eta}\, G_\varphi\, \eta} \tag{2.46}$$

with

$$(-\Delta + m_\phi^2)\cdot G_\phi = \delta^d(x-y)$$
$$(\not{\partial} + \underset{\overset{\shortparallel}{m}}{m_\varphi})\, G_\varphi = \delta^d(x-y) \tag{2.47}$$

In general

$$e^{-\int_x V\left(\frac{\delta}{\delta J}\right) + h\int_x \frac{\delta}{\delta \eta}\frac{\delta}{\delta J}\frac{\delta}{\delta \bar{\eta}}}$$
$$\to e^{-\int_x V\left(\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}, \frac{\delta}{\delta J}\right)} \tag{2.48}$$

$V(\bar{\psi}, \psi, \varphi) = V(\varphi) + \int_x h\bar{\psi}\varphi\psi$

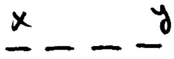
Feynman rules:

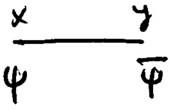
(a) Write down all diagrams in a given order N of the coupling

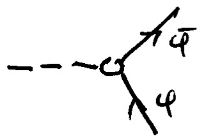
(b) Combinatorical factors

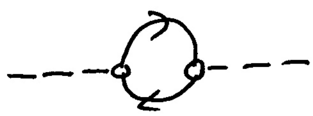
(c) (-1) for closed fermionic loops

Explanation for (c): Consider Yukawa theory

φ-prop:  : $G_\phi(x,y)$

ψ-prop:  : $G_\psi(x,y)$

Vertex:  : h

One-loop 2-point fct.: 

$$h^2 \left\langle T\, \varphi(x) \int_z \bar\psi_z \varphi_z \overbrace{\psi_z \int_{z'} \bar\psi_{z'} \varphi_{z'} \psi_{z'}}^{(-1)} \varphi(y) \right\rangle$$

$$= -h^2 \int_{z,z'} \left\langle T\, \varphi(x)\varphi(z) \left[\psi(z)\bar\psi(z')\right] \left[\psi(z')\bar\psi(z)\right] \varphi(z')\varphi(y) \right\rangle \tag{2.49}$$

Grassmann prop.

Via functional integral:

$$\langle T\varphi(x)\varphi(y)\rangle_{\text{connected}} = \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} \ln Z[\eta,\bar\eta,J]\Big|_{\eta,\bar\eta,J=0}$$

$$= \left[\frac{1}{Z}\frac{\delta^2 Z}{\delta\varphi(x)\,\delta\varphi(y)} - \left(\frac{1}{Z}\frac{\delta Z}{\delta\varphi(x)}\right)\left(\frac{1}{Z}\frac{\delta Z}{\delta\varphi(y)}\right)\right]_{\eta,\bar\eta,J=0} \tag{2.50}$$

$$\frac{\delta^2 Z}{\delta\varphi(x)\,\delta\varphi(y)} = \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)}\Big[1 + h\int_{x'} \frac{\delta}{\delta\eta(x')}\frac{\delta}{\delta\bar\eta(x')}\frac{\delta}{\delta J(x')}$$

$$(-1)$$

$$+ \frac{h^2}{2}\int_{x',x''} \frac{\delta}{\delta\eta(x')}\frac{\delta}{\delta\bar\eta(x')}\frac{\delta}{\delta J(x')}\,\frac{\delta}{\delta\eta(x'')}\frac{\delta}{\delta\bar\eta(x'')}\cdot\frac{\delta}{\delta J(x'')}$$

$$+\mathcal{O}(h^3)\Big]\, e^{\frac{1}{2}\int_{x,y} J\cdot G_\phi\cdot J - \int_{x,y}\bar\eta\, G_\psi\, \eta} \qquad (2.51)$$

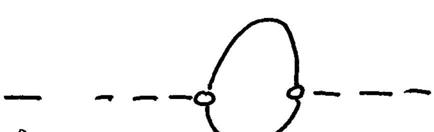
At vanishing fields the linear term (in h) vanishes, and the h^2-term gives the 1-Loop contr.

$$\Rightarrow \boxed{\left.\frac{\delta^2 Z}{\delta\varphi(x)\,\delta\varphi(y)}\right|_{1\text{-Loop}} = G_\phi(x,y) + \overset{(*)}{1}\, h^2\int_{x',x''}\Big\{ G_\phi(x,x')\,G_\phi(x'',y)\Big[(-)\,G_{\psi\,\xi'\xi''}(x',x'')\cdot G_{\psi\,\xi''\xi'}(x'',x')\Big]\Big\}} \qquad (2.52)$$

spin-indi.

minus sign of fermionic Loops

h^2-term:



$h^2/2$	$(\frac{1}{2}J G_\phi J)^2$	$(\bar\eta G_\psi \eta)^2$	ext. δ_J	int. δ_J	$\delta_\eta^2\,\delta_{\bar\eta}^2$	
(*) $\frac{1}{2}$	$\frac{1}{4}\cdot\frac{1}{2}$	$\cdot\,\frac{1}{2}$	2	$\cdot\; 4\cdot 2$	$\cdot\; 2$	$=1$

In momentum space:

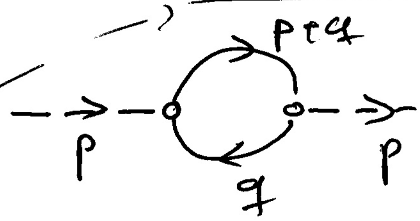
$$G_\psi(p) = \frac{1}{i\not p + m} = \frac{-i\not p + m}{p^2 + m^2}$$

Euclidean

$$G_\phi(p) = \frac{1}{p^2 + m^2} \qquad (2.53)$$

Euclidean

Vacuum polarisation:



$$\simeq -h^2 \int \frac{d^dq}{(2\pi)^d}\, \mathrm{tr}\left(-i\not q + m\right)\left(-i(\not p + \not q) + m\right) \qquad (2.54)$$

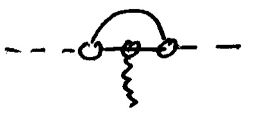
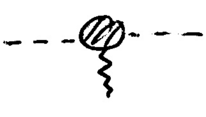
fermionic loop   Dirac trace

$$\cdot \frac{1}{q^2+m^2} \cdot \frac{1}{(p+q)^2+m^2}$$

$$\mathrm{tr}\, \gamma^\mu \gamma^\nu = d\, \delta^{\mu\nu}$$

$$= -h^2 \int \frac{d^dq}{(2\pi)^d} \frac{\left[-d\, q\cdot(p+q) + d\, m^2\right]}{\left(q^2+m^2\right)\left((p+q)^2+m^2\right)}$$

Exercise

Interpretation: ϕ is a neutral field, but gets an 'effective' dipole via virtual $\psi\bar\psi$-pairs (e.g. e^+e^-); in QED coupling to photon via  ~ , also