3 Functional Methods*

So far we have worked with the generating functional Z (partition function), which generates connected & disconnected correlation functions.

We have also introduced the Schwinger functional W,

$$W[J] := \ln Z[J] \qquad (3.1)$$

which generates connected Green functions (proof below).

3.1 Effective action

The effective action Γ is the generating functional of One-particle-irreducible (1PI) Green/Correlation functions (proof below).

It follows from W via a Legendre transformation: (scalar field)

$$\Gamma[\phi] := \sup_J \left[\int d^d x \, J(x)\,\phi(x) - W[J] \right] \quad (3.2)$$

We assume now that we have a maximum and differentiability (w.r.t. J, ϕ), and hence

$$\frac{\delta}{\delta J(y)} \left\{ \int_x J \cdot \phi - W[J] \right\} \Bigg|_{J = J_{max}} = 0$$

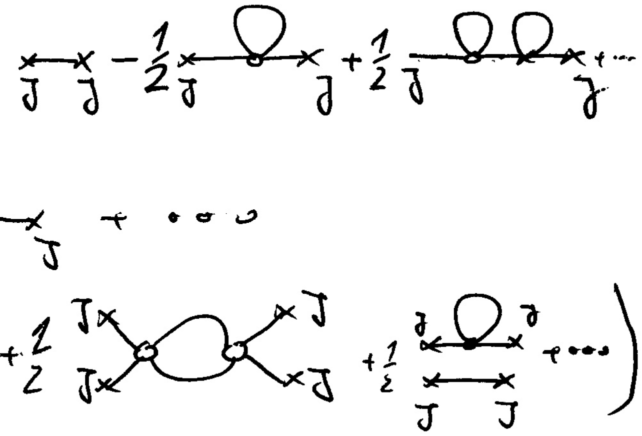
$$\Longrightarrow \quad \phi(y) = \frac{\delta W}{\delta J(y)}\bigg|_{J_{max}} = \frac{1}{Z[J_{max}]} \frac{\delta Z}{\delta J(y)}\bigg|_{J_{max}} \quad (3.3)$$

or $\boxed{\phi(y) = \langle \varphi(y) \rangle_{J_{max}}}$

The Legendre transform of Γ is W (strictly speaking the convex hull of W), and hence

$$\boxed{J(x) = \frac{\delta \Gamma}{\delta \phi(x)}} \quad (3.4)$$

Redundancies:

$Z[J]$ ~ vac. diagr. $\Big(1 +$  $\Big)$

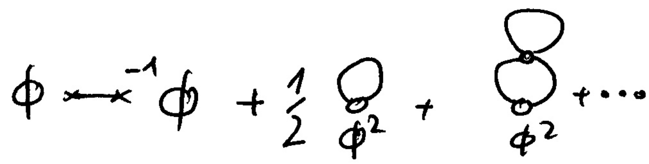
con. + discon.

$\Big\downarrow$ ln

$W[J] = \ln Z[J] \sim \ln(\text{vac. diagr.}) + \ln(1 + \cdots)$

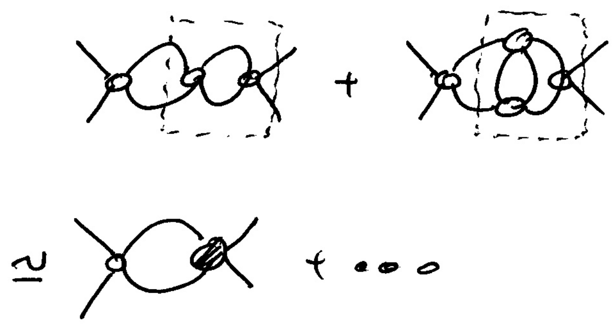
connected

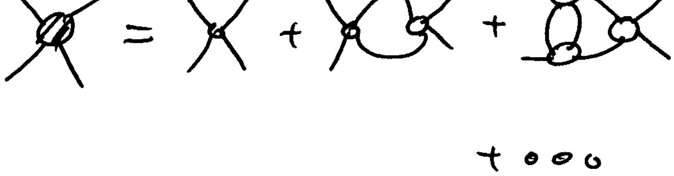
$\Big\downarrow$ Legendre

$\Gamma = \sup_J \left(\int_x J\cdot\phi - W[J] \right) \simeq$ 

1PI

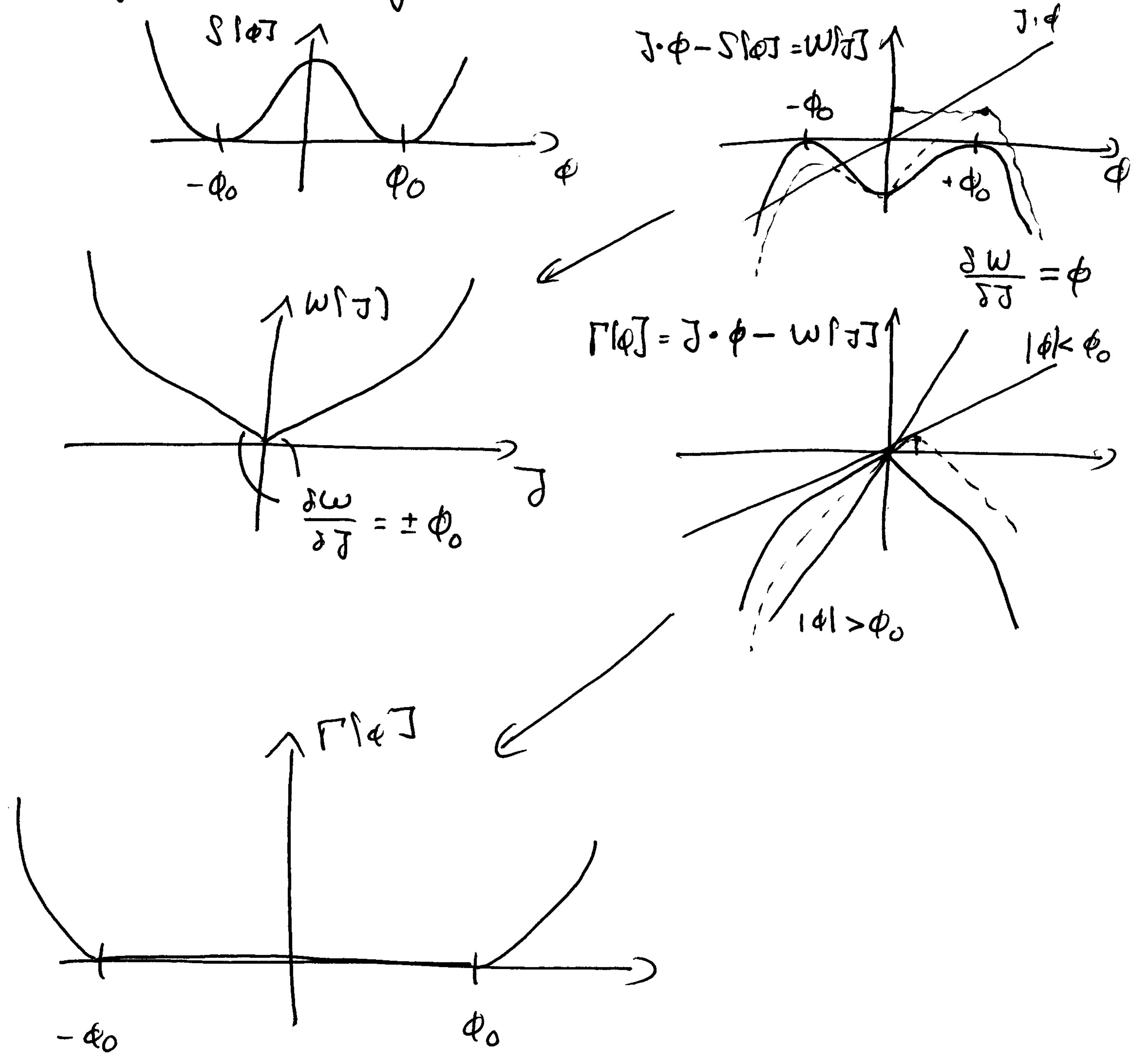
$\Big\downarrow$ DSE

Closed form: eg 

with 

$+ \cdots$

Legendre transform:



$\Gamma[\varphi]$ convex hull of $S[\varphi]$

$$\boxed{\frac{\delta^2 \Gamma}{\delta \phi^2} \geq 0}$$

What are the derivatives (moments) of Γ? We start with the observation, that

$$W^{(2)}[J]: \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = \frac{1}{Z[J]}\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)} - \frac{1}{Z[J]}\frac{\delta Z}{\delta J(x)}\frac{1}{Z[J]}\frac{\delta Z}{\delta J(y)}$$

$$= \underbrace{\langle \varphi(x)\varphi(y)\rangle_J - \overset{\text{disconnected 2 point}}{\langle\varphi(x)\rangle_J\langle\varphi(y)\rangle_J}}_{\text{connected}}$$

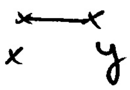
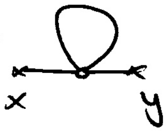
$$\underset{\text{full propagator}}{W^{(2)}[J]} = \langle\varphi(x)\varphi(y)\rangle_c := \qquad (3.5)$$

We conclude that $W^{(2)}$ is the connected 2-point function. What is $\Gamma^{(2)}[\phi] = \frac{\delta^2\Gamma[\phi]}{\delta\phi^2}$:

$$\delta^d(x-y) = \frac{\delta J(x)}{\delta J(y)} \underset{(3.4)}{=} \frac{\delta}{\delta J(y)}\frac{\delta\Gamma}{\delta\phi(x)} = \int_z \frac{\delta\phi(z)}{\delta J(y)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(x)}$$

$$= \int_z \frac{\delta^2 W[J]}{\delta J(y)\,\delta J(z)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(x)} \qquad (3.6)$$

$$\Longrightarrow \quad \boxed{\underbrace{\langle\varphi(x)\varphi(y)\rangle_c}_{W^{(2)''}(x,y)} = \frac{1}{\Gamma^{(2)}}(x,y)} \qquad (3.7)$$

$$W^{(2)}(x,y) = \langle \varphi(x)\varphi(y)\rangle_C =$$  $+\frac{1}{2}$  $-$  $+\cdots$

$$= \frac{1}{\Gamma^{(2)}}(x,y) = \frac{1}{\text{—}^{-1} - \frac{1}{2}\,8 + 2\text{-loop}}(x,y)$$

$$\parallel$$

$$\frac{\delta^2}{\delta\phi^2}\left[\frac{1}{2}\mathrm{Tr}\ln S^{(2)}\right]_{\phi=0}$$